ОСНОВИ НА МОДЕЛНОТО ПРЕДСКАЗВАЩО УПРАВЛЕНИЕ

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БЪЛГАРСКА
АКАДЕМИЯ
на НАУКИТЕ
1869

СЪДЪРЖАНИЕ

1. Въведение в теорията на оптималното управление.
2. Линейно квадратично регулиране при наличие на ограничения.
3. Моделно предсказващо управление на линейни системи с ограничения.
4. Робастно моделно предсказващо управление на линейни системи при наличие на ограничени по амплитуда смущения.
5. Моделно предсказващо управление на система от втори ред.
6. Моделно предсказващо управление на лабораторен сепаратор.

1. Introduction to optimal control theory

General optimal control problem formulation
(Ray, 1981)

System described by a set of non-linear, non-autonomous state equations:

$$\frac{dx}{dt} = f(x, u, t)$$

x - n-dimensional vector of state variables
u - m-dimensional vector of control variables
f - n-dimensional vector function
t - time

Initial state of the system:

$$x(0) = x_0$$

Set of constraints:

$$g(x,u) \geq 0$$

$$h(x,u) = 0$$

Admissible range of control variables:

$$u_{\min,i} \leq u_i \leq u_{\max,i} \quad , \quad i = 1, \ldots, m$$

Terminal constraint describing the target set:

$$\psi[x(t_f)] \geq 0$$

Performance index:

$$I[u(t)] = G[x(t_f)] + \int_0^{t_f} F(x,u)dt \quad \rightarrow \ \min$$

Special cases:

- ***minimum-time control problem***

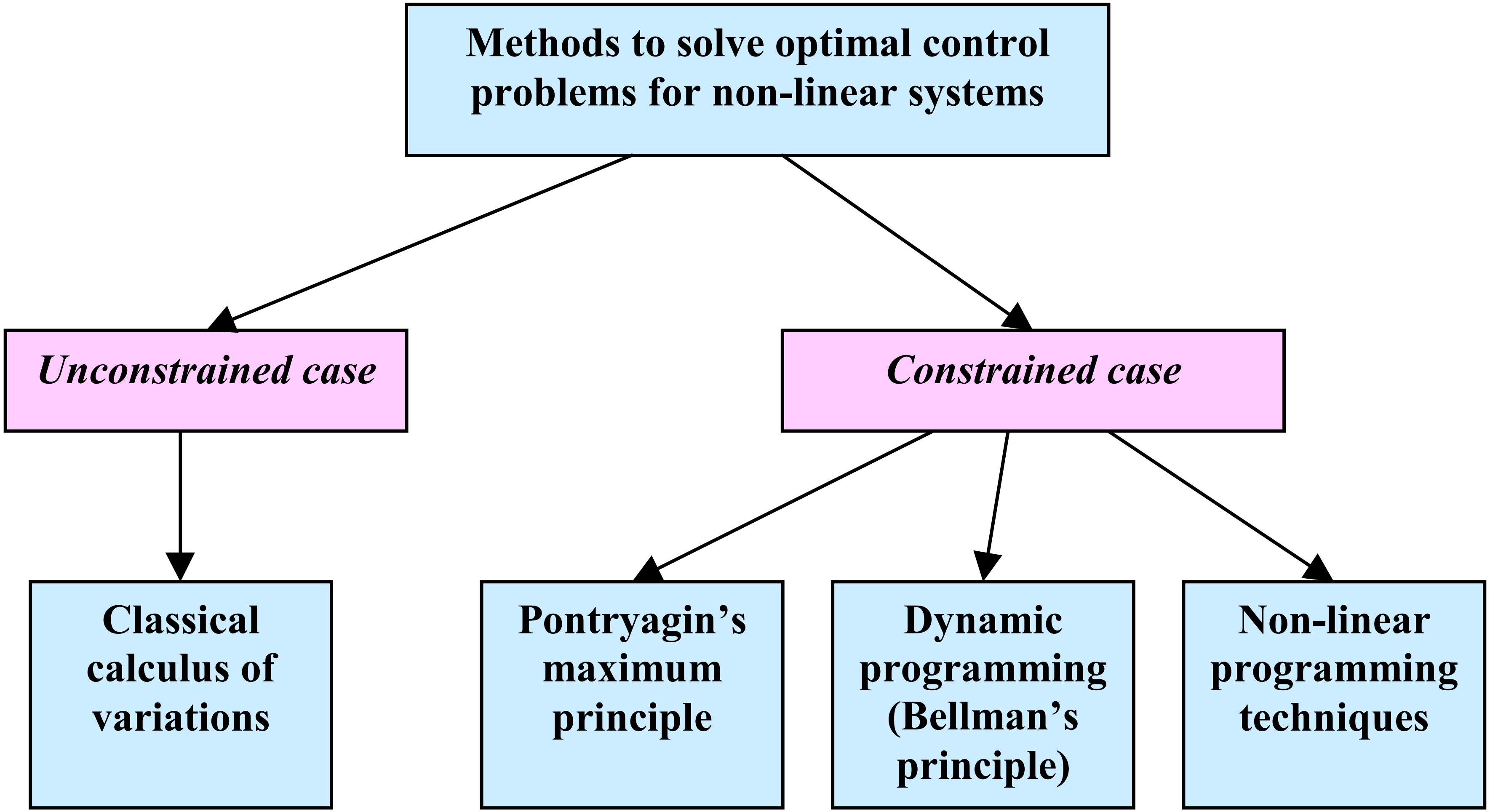
$$I[u(t)] = \int_0^{t_f} 1\,dt \quad \rightarrow \ \min$$

- ***maximal productivity problem***

$$I[u(t)] = G[x(t_f)] \quad \rightarrow \ \max$$

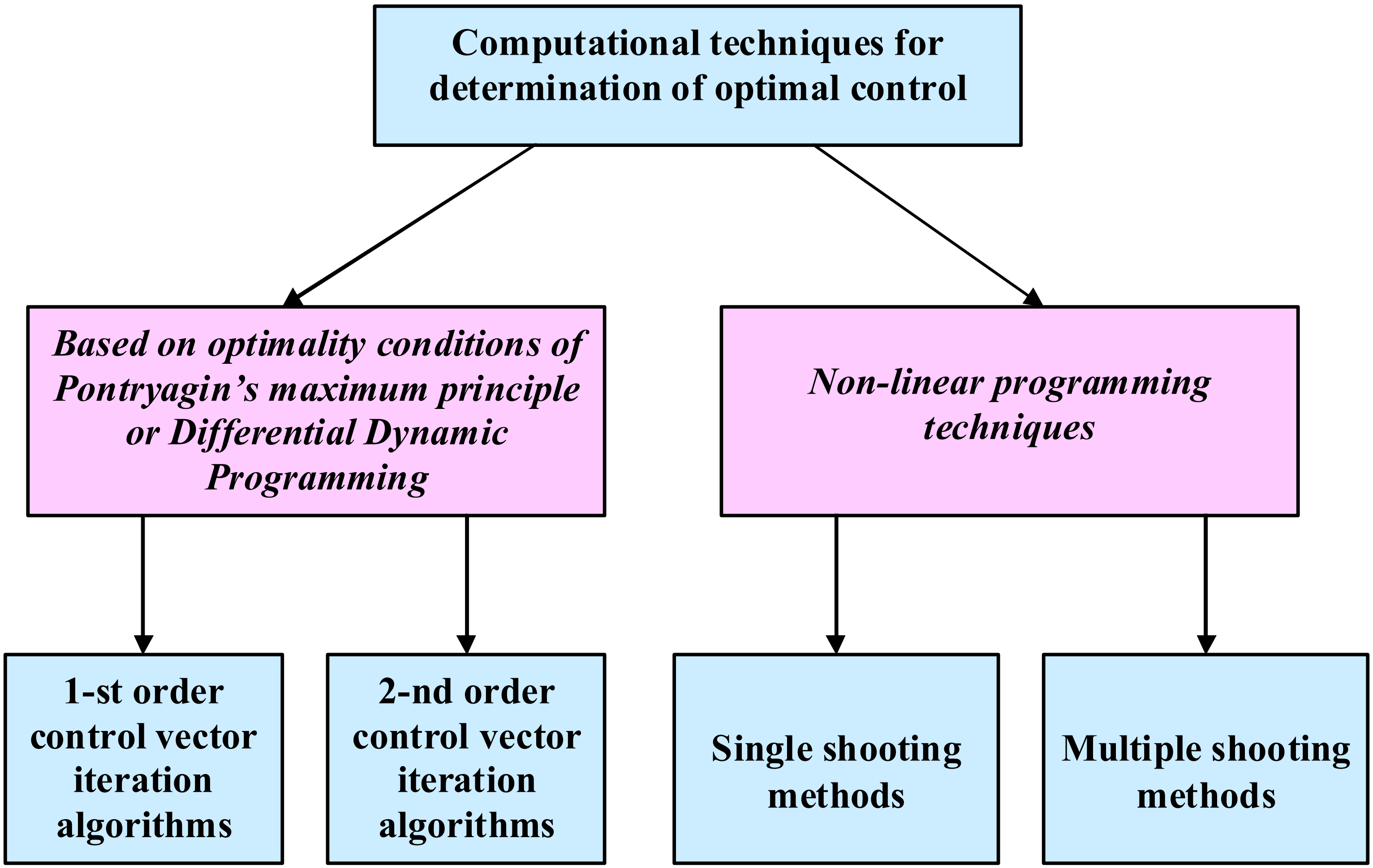
- ***minimize an integral criterion***

$$I[u(t)] = \int_0^{t_f} F(x,u)\,dt \quad \rightarrow \ \min$$


Methods to solve optimal control problems for non-linear systems
Unconstrained case
Constrained case
Classical calculus of variations
Pontryagin's maximum principle
Dynamic programming (Bellman's principle)
Non-linear programming techniques

Characteristics → **Methods ↓**	**Takes into account presence of constraints**	**Mathematical model**	**Numerical difficulties**
Classical calculus of variations	No	continuous	TPBVP, stiffness
Pontryagin's maximum principle	Yes	continuous, discrete	TPBVP, stiffness
Dynamic programming (Bellman's principle)	Yes	continuous, discrete	curse of dimensionality
Non-linear programming techniques	Yes	continuous, discrete	high dimensional optimization problems

(Grancharova and Johansen, 2005)



(Grancharova and Johansen, 2005)

Control vector iteration algorithms

(based on the optimality conditions of Pontryagin's maximum principle)

1-st order methods :

(Ray, 1981)

$$u^{i+1}(t) = u^{i}(t) - \varepsilon \frac{\partial H}{\partial u} \quad , \quad \varepsilon > 0 , \ t \in [0; t_f]$$

$$H = F(x,u) + \lambda^T f(x,u)$$

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x} \ , \ \lambda_i(t_f) = \frac{\partial G}{\partial x_i} \text{ , for } x_i \text{ unspecified at } t_f$$

2 - nd order methods :

(Jacobson, 1968)

$$u^{i+1}(t) = u^{i}(t) - \left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\varepsilon\left(\frac{\partial H}{\partial u} + \left(\frac{\partial f}{\partial u}\right)^{T} s\right) + \left(\frac{\partial^2 H}{\partial u \partial x} + \left(\frac{\partial f}{\partial u}\right)^{T} P\right)\delta x\right]$$

$$0 < \varepsilon \leq 1,\ t \in [0; t_f]$$

$$-\frac{ds}{dt} = \left(\frac{\partial f}{\partial x}\right)^{T} s - \left(\frac{\partial^2 H}{\partial u \partial x} + \left(\frac{\partial f}{\partial u}\right)^{T} P\right)^{T}\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left(\frac{\partial H}{\partial u} + \left(\frac{\partial f}{\partial u}\right)^{T} s\right)$$

$$-\frac{dP}{dt} = \frac{\partial^2 H}{\partial x^2} + \left(\frac{\partial f}{\partial x}\right)^{T} P + P\frac{\partial f}{\partial x} - \left(\frac{\partial^2 H}{\partial u \partial x} + \left(\frac{\partial f}{\partial u}\right)^{T} P\right)^{T}\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left(\frac{\partial^2 H}{\partial u \partial x} + \left(\frac{\partial f}{\partial u}\right)^{T} P\right)$$

with boundary conditions at t_f: $P = \frac{\partial^2 G}{\partial x^2}$, $s = 0$

Single shooting methods:

control vector parameterization by a set of trial functions of time:

(Ray, 1981)

$$u_i(t) = \sum_{j=1}^{l} a_{ij}\phi_{ij}(t)$$

control vector parameterization by a set of trial functions of state:

(Ray, 1981)

$$u_i(t) = \sum_{j=1}^{l} b_{ij}\phi_{ij}(x)$$

Multiple shooting methods:

(approximation of both control and state variables)

(Strand, 1989)

Parameters to be optimized:

$$P = [x_1(t_{ij}), x_2(t_{ij}), \ldots, x_n(t_{ij}), u_1(t_j), u_2(t_j), \ldots, u_m(t_j)]$$

$$j = 1, 2, \ldots, M \;,\; i = 1, 2, \ldots, N_j \;,\; t_{ij} \in [t^{j-1}; t^j]$$

2. Constrained linear quadratic regulation

(Scokaert and Rawlings, 1998)

Linear time-invariant discrete-time system:

$$x(t+k+1) = Ax(t+k) + Bu(t+k) \quad , \quad k \geq 0$$

Performance index:

$$I\left[x(t), \{u(t), u(t+1), \ldots\}\right] =$$

$$\sum_{k=0}^{\infty} \left[x^T(t+k)Qx(t+k) + u^T(t+k)Ru(t+k)\right]$$

Constraints:

$$Hx(t+k+1) \leq h \quad , \quad k \geq 0$$

$$Gu(t+k) \leq g \quad , \quad k \geq 0$$

Problem 1 – Unconstrained linear quadratic regulation

$$\min_{\{u(t),u(t+1),\ldots\}} \sum_{k=0}^{\infty} \left[x^T(t+k) Q x(t+k) + u^T(t+k) R u(t+k) \right]$$

subject to:

$$x(t+k+1) = Ax(t+k) + Bu(t+k) \quad , \quad k \geq 0$$

Solution:

$$u(t+k) = -Kx(t+k) \quad , \quad k \geq 0$$

$$K = (B^T P B + R)^{-1} B^T P A$$

Discrete-time algebraic Riccati equation:

$$P = A^T P A - A^T P B (B^T P B + R)^{-1} (A^T P B)^T + Q$$

Problem 2 - Constrained linear quadratic regulation:

$$\min_{\{u(t),u(t+1),\ldots\}} \sum_{k=0}^{\infty} \left[x^T(t+k)Qx(t+k) + u^T(t+k)Ru(t+k) \right]$$

subject to:

$$x(t+k+1) = Ax(t+k) + Bu(t+k) \quad , \quad k \geq 0$$

$$Hx(t+k+1) \leq h \quad , \quad k \geq 0$$

$$Gu(t+k) \leq g \quad , \quad k \geq 0$$

difficulty - infinite number of decision variables in the optimization and infinite number of constraints

Problem 3 - Model Predictive Control (MPC) Problem:

$$\min_{\{u(t),u(t+1),\ldots u(t+N-1)\}} \sum_{k=0}^{N-1} \left[x^T(t+k)Qx(t+k) + u^T(t+k)Ru(t+k) \right]$$

subject to:

$$x(t+k+1) = Ax(t+k) + Bu(t+k) \quad , \quad k \geq 0$$

$$Hx(t+k+1) \leq h \quad , \quad k = 0,1, \ldots, N-1$$

$$Gu(t+k) \leq g \quad , \quad k = 0,1, \ldots, N-1$$

$$u(t+k) = -Kx(t+k) \quad , \quad k \geq N$$

- finite number of decision variables, Nm, and finite number of constraints $N(n_h+n_g)$;
- it can be solved with standard quadratic programming methods

- **The finite-horizon optimization (Model Predictive Control)** also provides the solution to the **infinite-horizon linear quadratic regulation problem with constraints** (Sznaier and Damborg, 1987)

- For any given compact set of initial conditions, the algorithm in (Chmielewski and Manousiouthakis, 1996) provides **the horizon *N*** such that the **MPC controller** (solution of Problem 3) **solves the infinite horizon problem** (Problem 2)

Implicit approach to constrained linear quadratic regulation

implicit - the optimal control does not have the form of feedback control law, but it is obtained in the form of open-loop time trajectory

<u>Definition:</u>

Let $X_K \subseteq R^n$ denotes the set of states $x(t)$ for which the unconstrained LQR law, $u(t + k) = - Kx(t + k)$, $k \geq 0$, satisfies all constraints.

Control Algorithm (Scokaert and Rawlings, 1998):

0). Choose a finite horizon N_0, set $N = N_0$.

1). Solve *Problem 3* (*MPC problem*).

2). If $x(t + N) \in X_K$, go to step 4).

3). Increase N, go to step 1).

4). Terminate: $\pi^* = \pi_N$.

$\pi_N = [u(t), u(t+1), \ldots u(t+N-1)]$ – the optimal control trajectory determined by solving the MPC problem

π^* – the optimal control trajectory that is a solution of the constrained LQR problem

3. Model Predictive Control of Constrained Linear Systems

- **MPC problem formulation**

(Mayne et al., 2000; Bemporad et al., 2002)

Consider the problem of regulating to the origin the discrete-time linear time invariant system:

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) \quad (1)$$

while satisfying the following constraints:

$$y_{\min} \le y(t) \le y_{\max} \quad (2)$$

$$u_{\min} \le u(t) \le u_{\max} \quad (3)$$

Model Predictive Control (MPC) solves the problem in the following way:

Assume that a full measurement of the state $x(t)$ is available at the current time t. Then, the optimization problem:

$$\min_{U \equiv \{u_t, u_{t+1}, \ldots u_{t+N_u-1}\}} \left\{ \begin{array}{l} I[U, x(t)] = x^T_{t+N_y|t} P x_{t+N_y|t} + \\ \qquad \sum_{k=0}^{N_y-1} \left[x^T_{t+k|t} Q x_{t+k|t} + u^T_{t+k} R u_{t+k} \right] \end{array} \right\} \qquad (4)$$

subject to:

$$x_{t|t} = x(t)$$

$$y_{\min} \le y_{t+k|t} \le y_{\max} \quad , \quad k = 1, \ldots, N_c$$

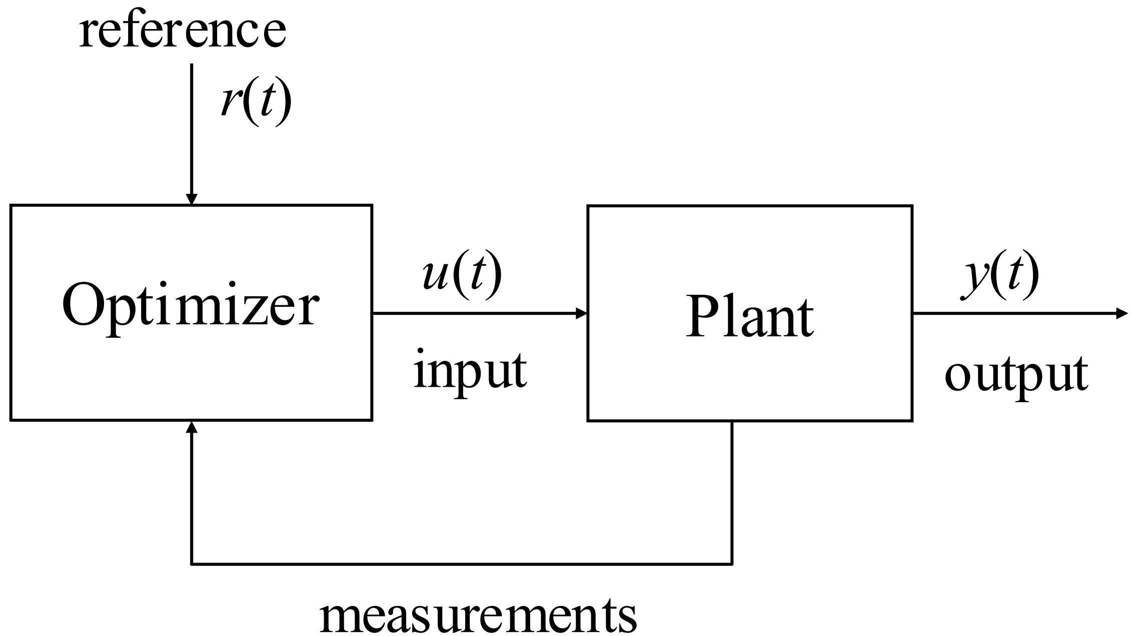
$$u_{\min} \le u_{t+k} \le u_{\max} \quad , \quad k = 0, 1, \ldots, N_c$$

$$x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k} \quad , \quad k \ge 0$$

$$y_{t+k|t} = Cx_{t+k|t} \quad , \quad k \ge 0$$

$$u_{t+k} = -Kx_{t+k|t} \quad , \quad N_u \le k < N_y$$

is solved at each time t.



N_y - ouput horizon (prediction horizon)

N_u - input horizon

N_c - constraints horizon

$N_u \leq N_y\,,\;\; N_c \leq N_y$ - 1

1). Choose $K=0$ and P as the solution of the Lyapunov equation:

$$P = A^T P A + Q \qquad (5)$$

2). Set $K=K_{\mathrm{LQ}}$, where K_{LQ} and P are the solution of the unconstrained infinite horizon LQR problem with weights Q and R:

$$K_{LQ} = \left(R + B^T P B\right)^{-1} B^T P A \qquad (6)$$

$$P = \left(A + BK_{LQ}\right)^T P \left(A + BK_{LQ}\right) + K_{LQ}{}^T R K_{LQ} + Q$$

Moving (receding horizon) strategy:

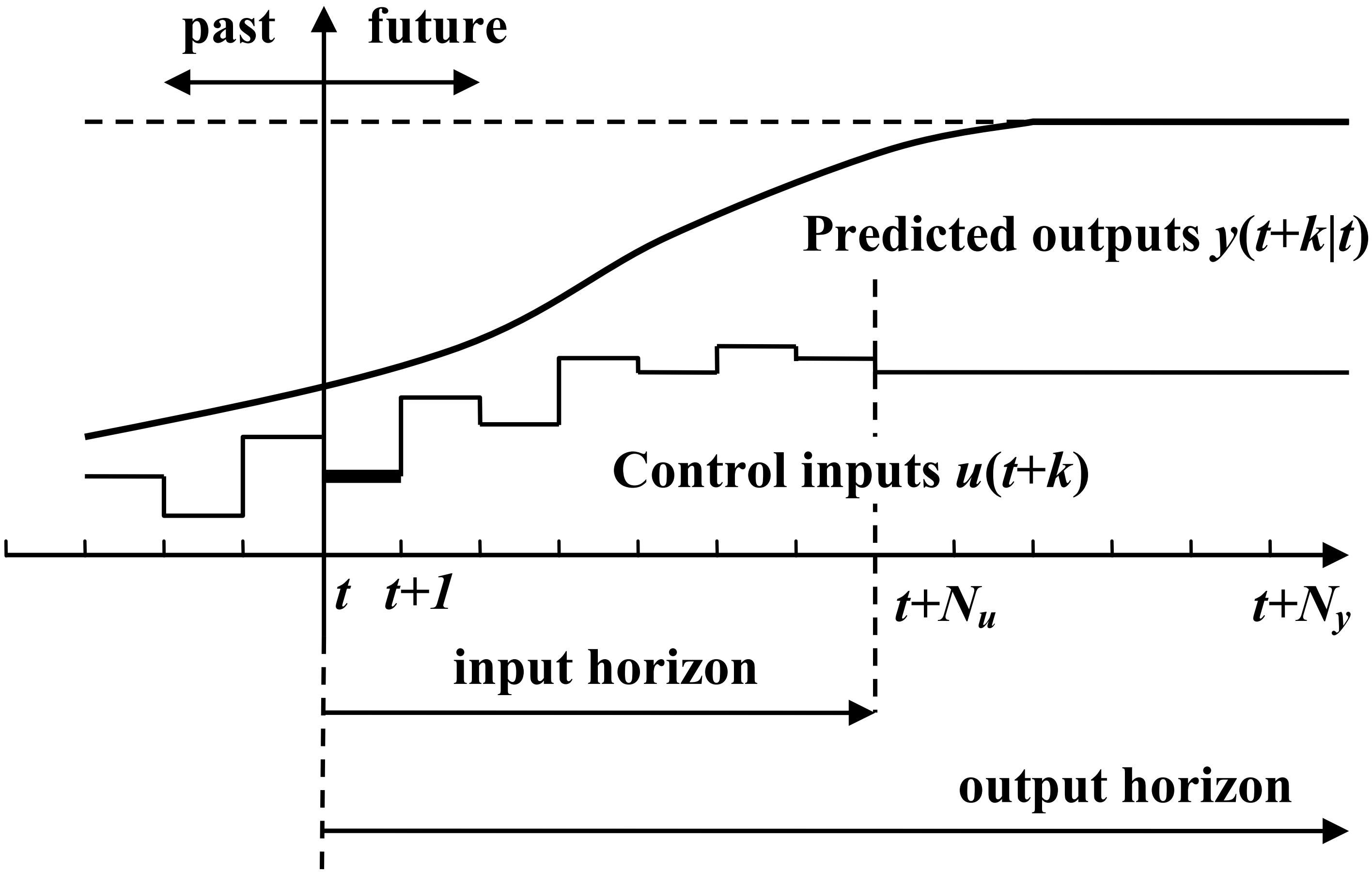
At time t compute the optimal solution

(the optimal input sequence):

$$U^*(t) = [u_t^*, \dots, u_{t+N_u-1}^*]$$

and apply to the system only the first input from the sequence:

$$u(t) = u_t^* \qquad (7)$$



Receding horizon strategy: only the first input of the computed optimal input sequence is implemented

- **Stability**

Theorem (Bemporad et al., 2002):

Let $N_y = \infty$, $K = 0$ or $K = K_{LQ}$, and $N_c < \infty$ be sufficiently large for guaranteeing existence of feasible input sequences at each time step. Then the MPC law (4) - (7) asymptotically stabilizes system (1) while enforcing the fulfillment of the constraints (2) - (3) from all initial states $x(0)$ such that the optimization problem (4) is feasible at $t = 0$.

(For more details about stability, see (Mayne et al., 2000))

▪ MPC computation

By substituting:

$$x_{t+k|t} = A^k x(t) + \sum_{j=0}^{k-1} A^j B u_{t+k-1-j}$$

the optimization problem becomes:

$$V[x(t)] = \frac{1}{2} x^T(t) Y x(t) + \min_U \left\{ \frac{1}{2} U^T H U + x^T(t) F U \right\}$$

subject to:

$$GU \leq W + Ex(t)$$

Quadratic Programming problem

$U \equiv [u_t^T, \ldots, u_{t+N_u-1}^T]^T \in R^s$, $s = mN_u$ – the optimization vector

4. Robust Model Predictive Control of Constrained Linear Systems With Bounded Disturbances

(Grancharova and Johansen, 2005)

Linear system with ***disturbance input***:

$$x(t+1) = Ax(t) + Bu(t) + T\theta(t),\ \theta(t) \in \Theta^A \subset R^s$$
$$y(t) = Cx(t)$$

Disturbance realization:

$$\Theta \equiv \left[\theta_t^T, \ldots, \theta_{t+N-1}^T\right]^T \in \Theta^B$$
$$\Theta \in \Theta^B = \left\{\Theta^A \times \Theta^A \ldots \times \Theta^A\right\} \subset R^{sN}$$

Optimization problem:

$$V^*(x(t),\Theta) = \min_{U \equiv \{u_t, \dots, u_{t+N-1}\}} J(U, x(t), \Theta)$$

Constraints:

$$u_{\min} \le u_{t+k} \le u_{\max},\ k = 0, 1, \dots, N-1$$

$$y_{\min} \le y_{t+k|t} \le y_{\max},\ k = 1, \dots, N$$

Cost function:

$$J(U, x(t), \Theta) = \sum_{k=0}^{N-1} \left[x_{t+k|t}^T Q x_{t+k|t} + u_{t+k}^T R u_{t+k} \right] + x_{t+N|t}^T P x_{t+N|t}$$

Nominal optimization criterion:

$$V_{nom}^{*}(x(t)) = \min_{U \equiv \{u_t, \ldots, u_{t+N-1}\}} J(U, x(t), \theta^N)$$

Nominal value
of the disturbance input:

$$\theta(t) = \theta^N = 0$$

Robustness:

Satisfaction of the output and input constraints under ***all possible disturbance realizations*** $\boldsymbol{\Theta} \in \boldsymbol{\Theta}^{\boldsymbol{B}}$

The robust MPC can be represented as a QP problem:

$$\boldsymbol{V}^{*}_{z,nom}(\boldsymbol{x}) = \min_{z} \ \frac{1}{2}\boldsymbol{z}^{T}\boldsymbol{H}\boldsymbol{z}$$

$$\text{subject to : } \boldsymbol{G}\boldsymbol{z} \le \boldsymbol{W} + \boldsymbol{S}_1\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{S}_2\boldsymbol{\Theta}, \ \forall \boldsymbol{\Theta} \in \boldsymbol{\Theta}^{\boldsymbol{B}}$$

Optimization variables:

$$\boldsymbol{z} \equiv \boldsymbol{U} + \boldsymbol{H}^{-1}\boldsymbol{F}^{T}\boldsymbol{x}(\boldsymbol{t}),\ \boldsymbol{U} \equiv \left[\boldsymbol{u}_{t}^{T}, \ldots, \boldsymbol{u}_{t+N-1}^{T}\right]^{T}$$

Assumption:

The disturbance input set $\boldsymbol{\Theta}^A = \{\boldsymbol{\theta} \in \boldsymbol{R}^s \mid \boldsymbol{\theta}^L \leq \boldsymbol{\theta} \leq \boldsymbol{\theta}^U\}$ represents a hyper-rectangle that includes the origin in its interior.

Definition:

Consider the ***i***-th constraint defined by $\boldsymbol{G}^{\boldsymbol{i}}, \boldsymbol{W}^{\boldsymbol{i}}, \boldsymbol{S}_1^{\boldsymbol{i}}, \boldsymbol{S}_2^{\boldsymbol{i}}$ rows of the matrices $\boldsymbol{G}, \boldsymbol{W}, \boldsymbol{S}_1, \boldsymbol{S}_2$. The worst disturbance realization for the ***i***-th constraint, denoted by $\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{i}} \in \boldsymbol{\Theta}^{\boldsymbol{B}}$, is one which solves the linear program:

$$\boldsymbol{S}_2^{\boldsymbol{i}} \widetilde{\boldsymbol{\Theta}}^{\boldsymbol{i}} = \min_{\boldsymbol{\Theta} \in \boldsymbol{\Theta}^{\boldsymbol{B}}} \{\boldsymbol{S}_2^{\boldsymbol{i}} \boldsymbol{\Theta}\}$$

Lemma:

If there exist a $\boldsymbol{z}$ *that satisfies the following constraint:*

$$\boldsymbol{Gz} \le \widetilde{\boldsymbol{W}} + \boldsymbol{S}_1 \boldsymbol{x}(\boldsymbol{t}),$$

where the $\boldsymbol{i}$*-th row of the matrix* $\widetilde{\boldsymbol{W}}$ *is determined by:*

$$\widetilde{\boldsymbol{W}}^{\boldsymbol{i}} = \boldsymbol{W}^{\boldsymbol{i}} + \boldsymbol{S}_2^{\boldsymbol{i}} \widetilde{\boldsymbol{\Theta}}^{\boldsymbol{i}},$$

and where $\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{i}} \in \boldsymbol{\Theta}^{\boldsymbol{B}}$ *is the worst disturbance realization for the* $\boldsymbol{i}$*-th constraint, then this implies that* $\boldsymbol{z}$ *will satisfy the constraint* $\boldsymbol{Gz} \le \boldsymbol{W} + \boldsymbol{S}_1 \boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{S}_2 \boldsymbol{\Theta}$ *for all possible disturbance realizations* $\boldsymbol{\Theta} \in \boldsymbol{\Theta}^{\boldsymbol{B}}$*. Such a control is referred to as* ***robustly feasible.***

Equivalent QP problem without disturbance:

$$V^*_{z,nom}(x) = \min_z \frac{1}{2} z^T H z$$

subject to:

$$Gz \leq \widetilde{W} + S_1 x(t)$$

5. Model Predictive Control of a Second-Order System

- **absence of disturbance**

$$x(t+1) = Ax(t) + Bu(t)$$

$$A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} T_s^2 \\ T_s \end{bmatrix}$$

Sampling time T_s=0.05 [s]

$$N = 2, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1$$

Constraints :

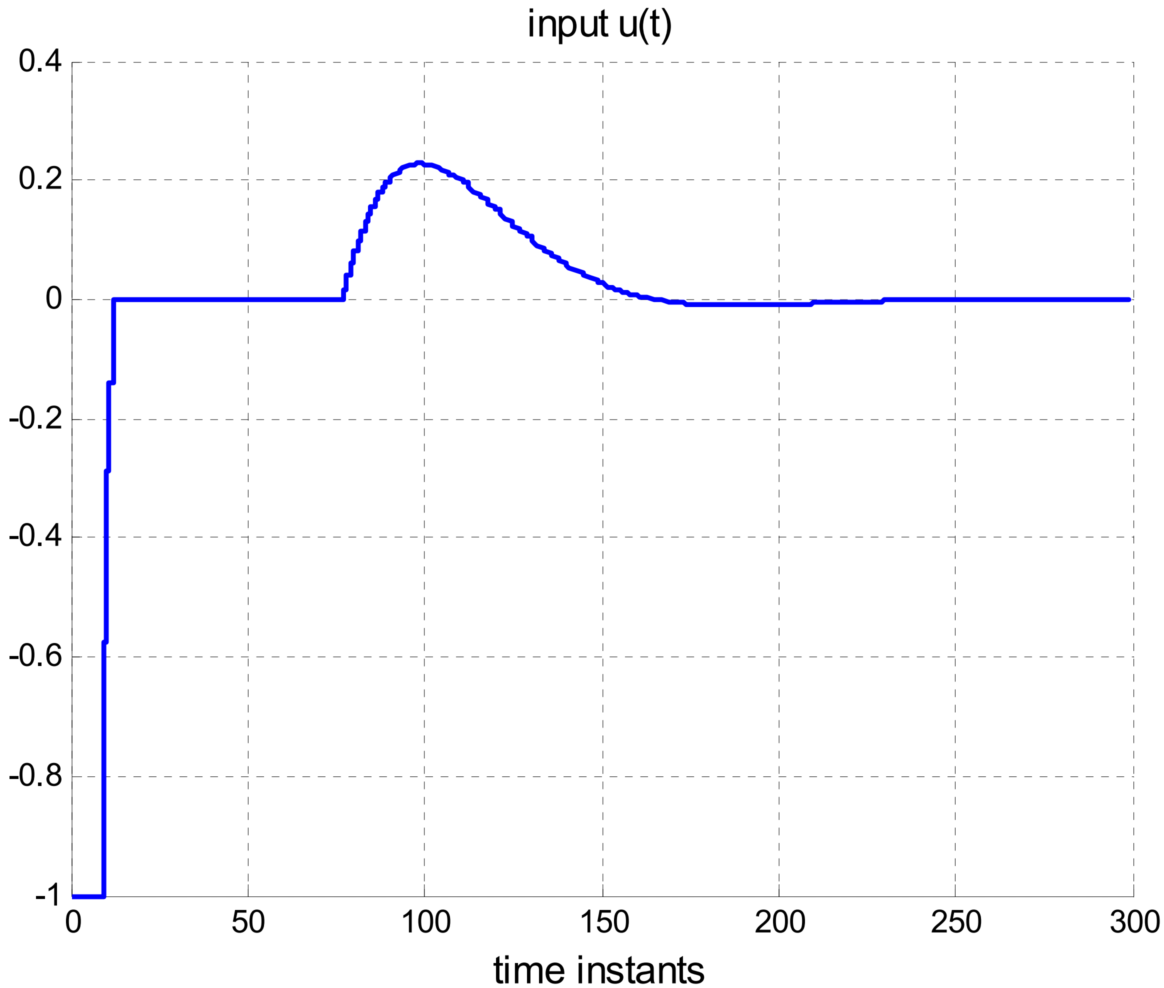
$$-1 \le u \le 1$$

$$-0.5 \le x_2 \le 0.5$$

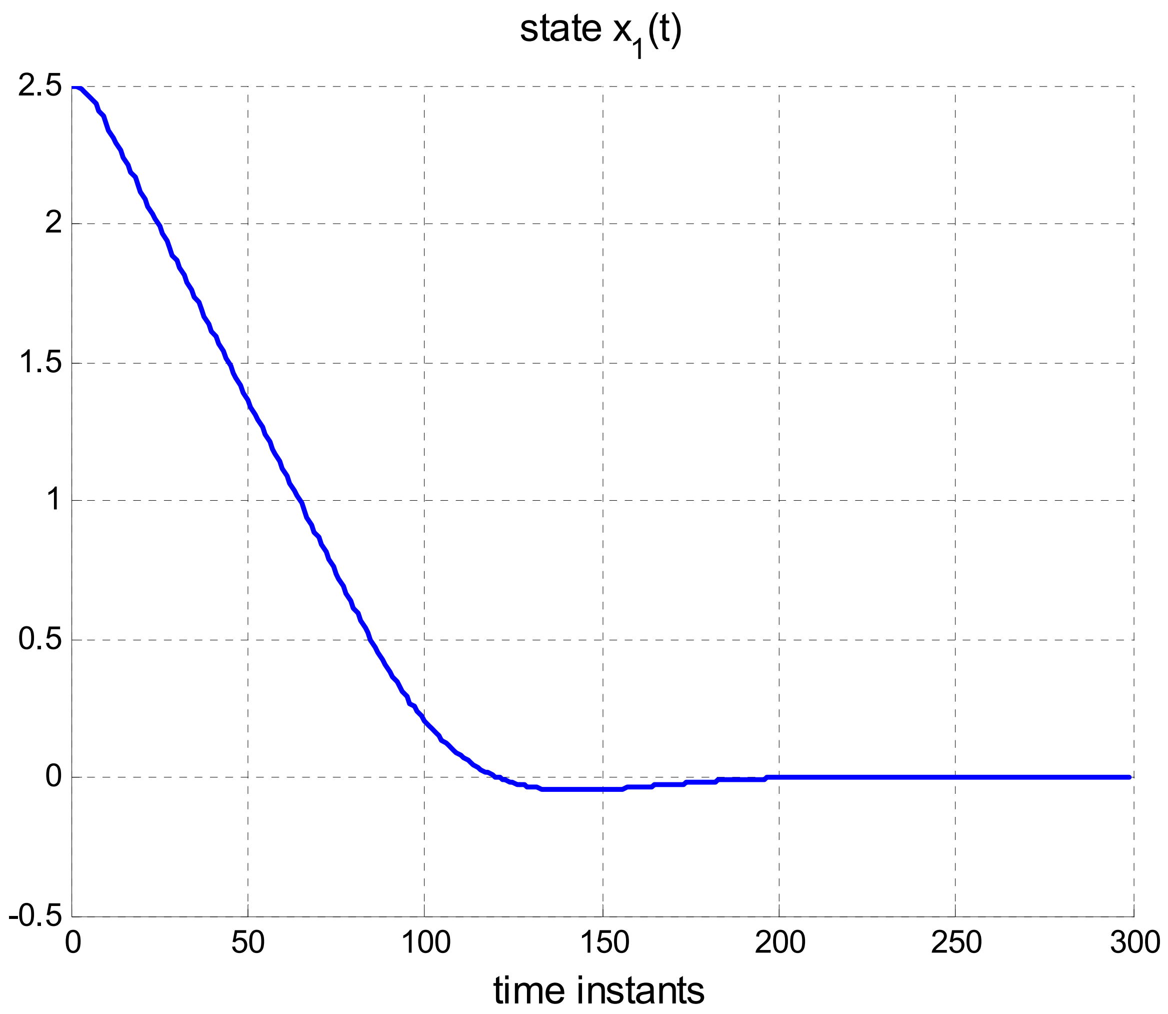
Matrices in the QP problem:

$$H = \begin{bmatrix} 2.157 & 0.152 \\ 0.152 & 2.147 \end{bmatrix} \qquad F = \begin{bmatrix} 2.218 & 2.072 \\ 3.146 & 3.035 \end{bmatrix}$$

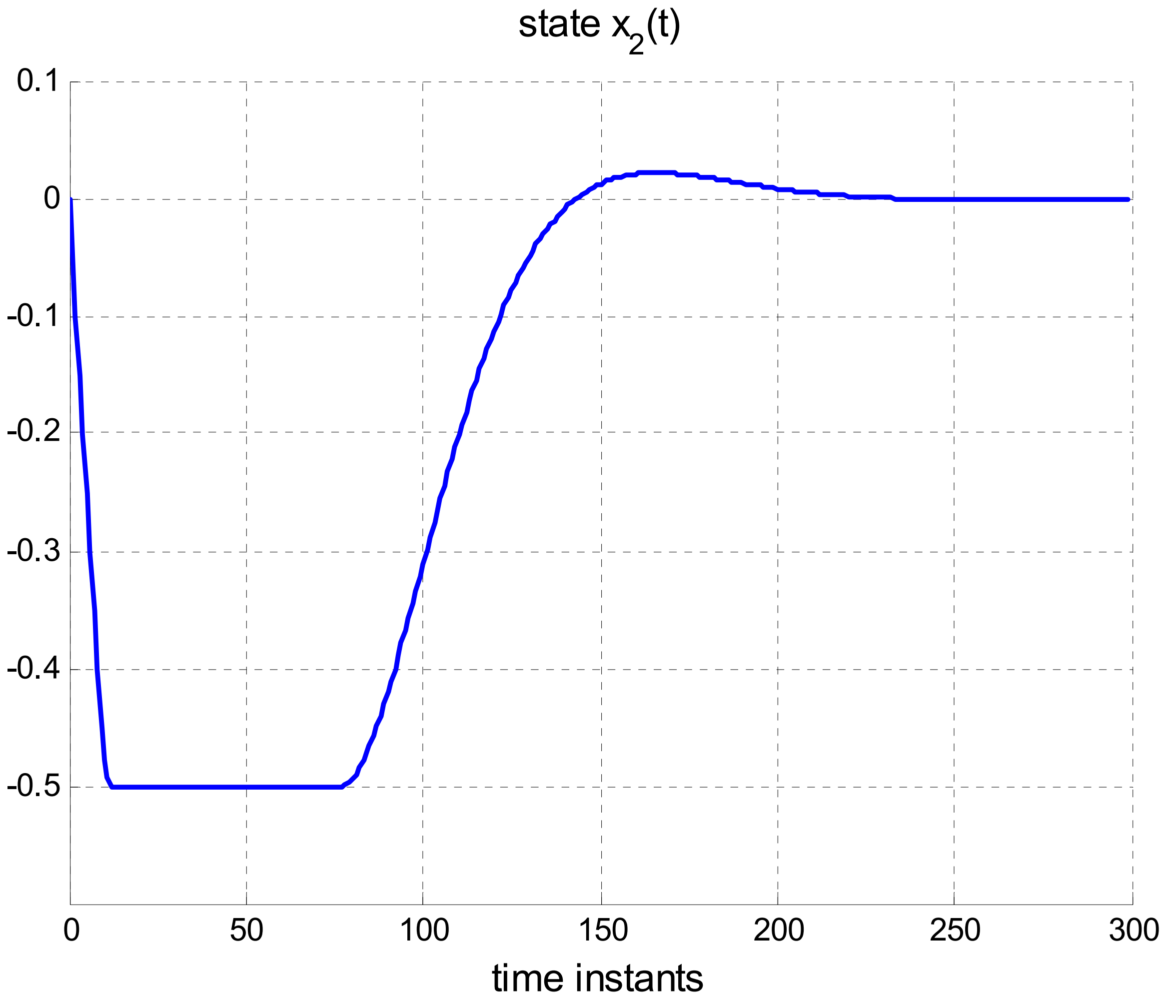
$$G = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -0.05 & 0 \\ -0.05 & -0.05 \\ 0.05 & 0 \\ 0.05 & 0.05 \end{bmatrix} \qquad W = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & -1 \end{bmatrix}$$



Optimal control input trajectory



Optimal trajectory of x_1



Optimal trajectory of x_2

- **presence of disturbance**

$$x(t+1) = Ax(t) + Bu(t) + T\theta(t)$$

$$A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} T_s^2 \\ T_s \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$N = 5, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1$$

Constraints:

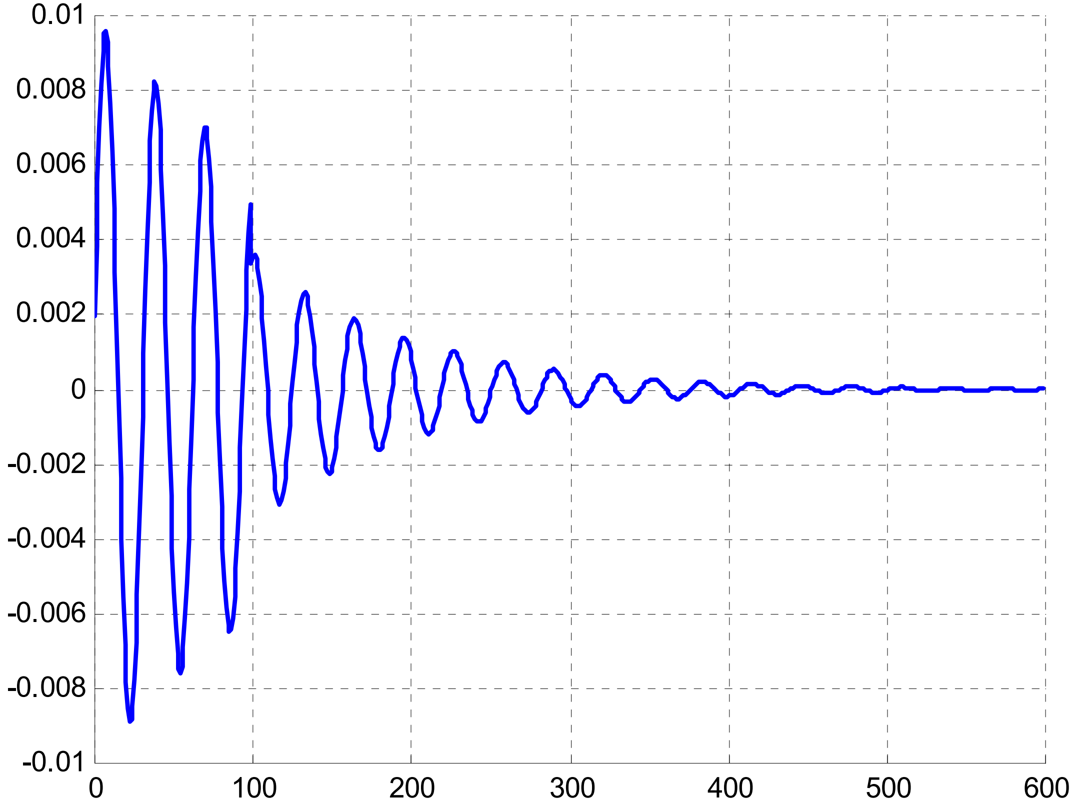
$$-1 \le u \le 1$$

$$-0.5 \le x_2 \le 0.5$$

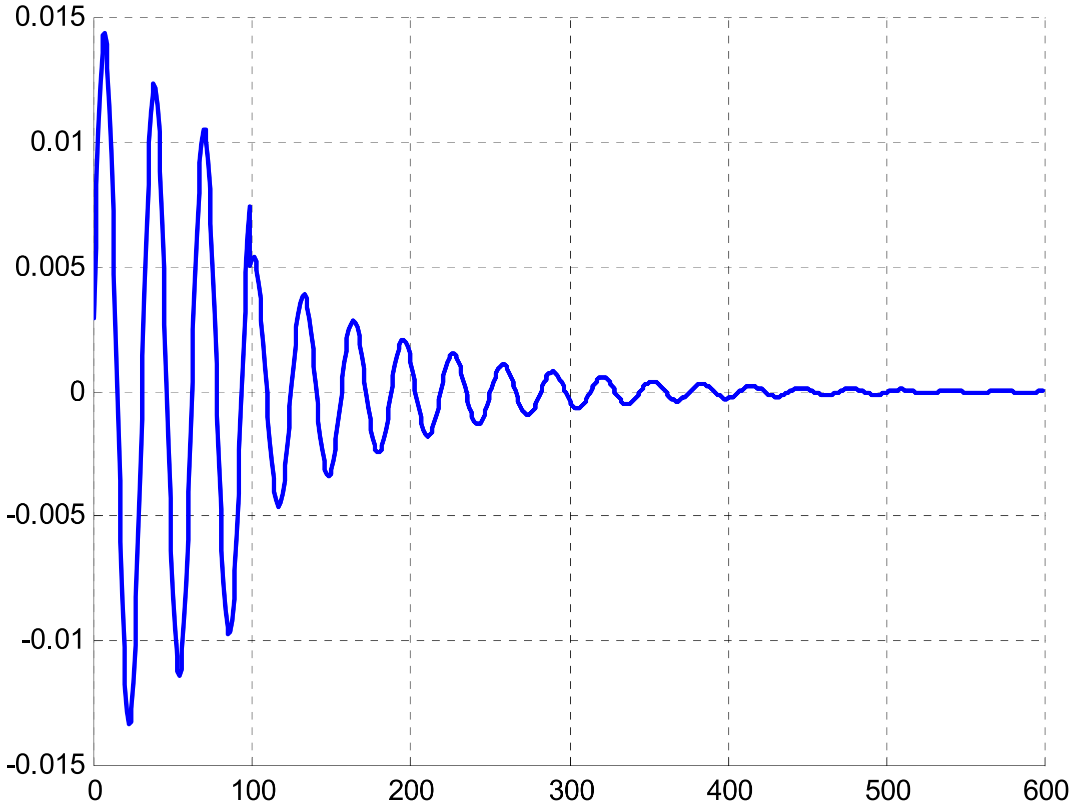
Disturbances have the following bounds:

$$-0.01 \le \boldsymbol{\theta}_1(\boldsymbol{t}) \le 0.01$$

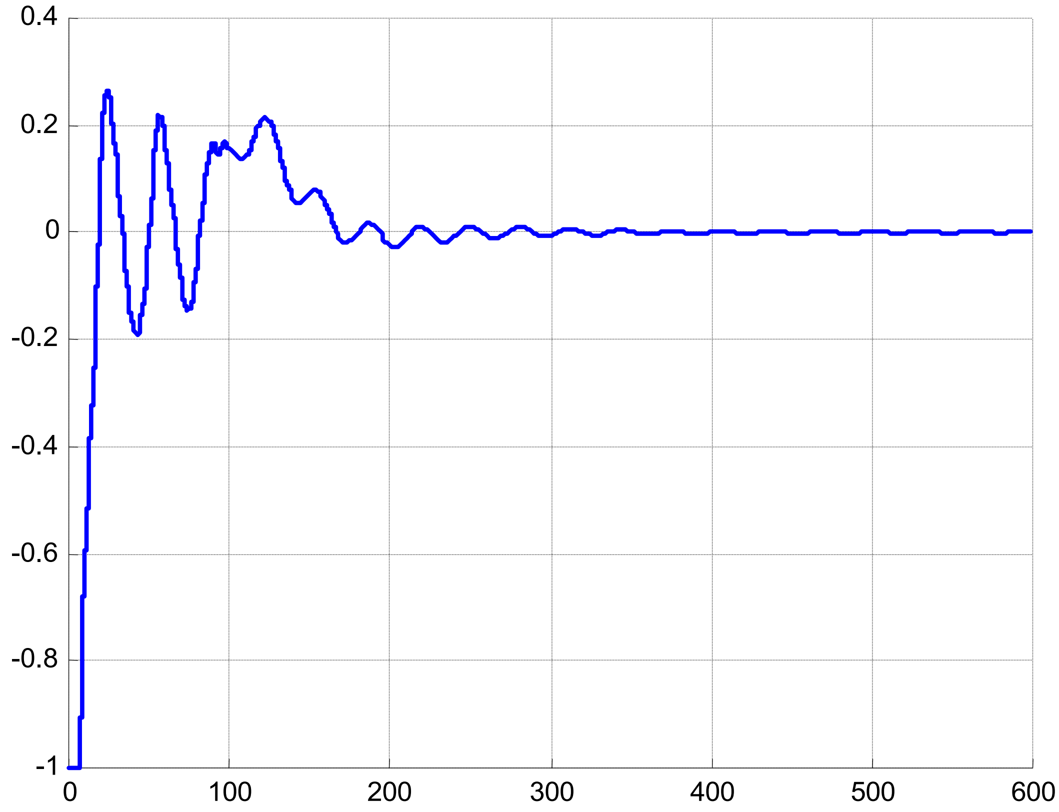
$$-0.015 \le \boldsymbol{\theta}_2(\boldsymbol{t}) \le 0.015$$



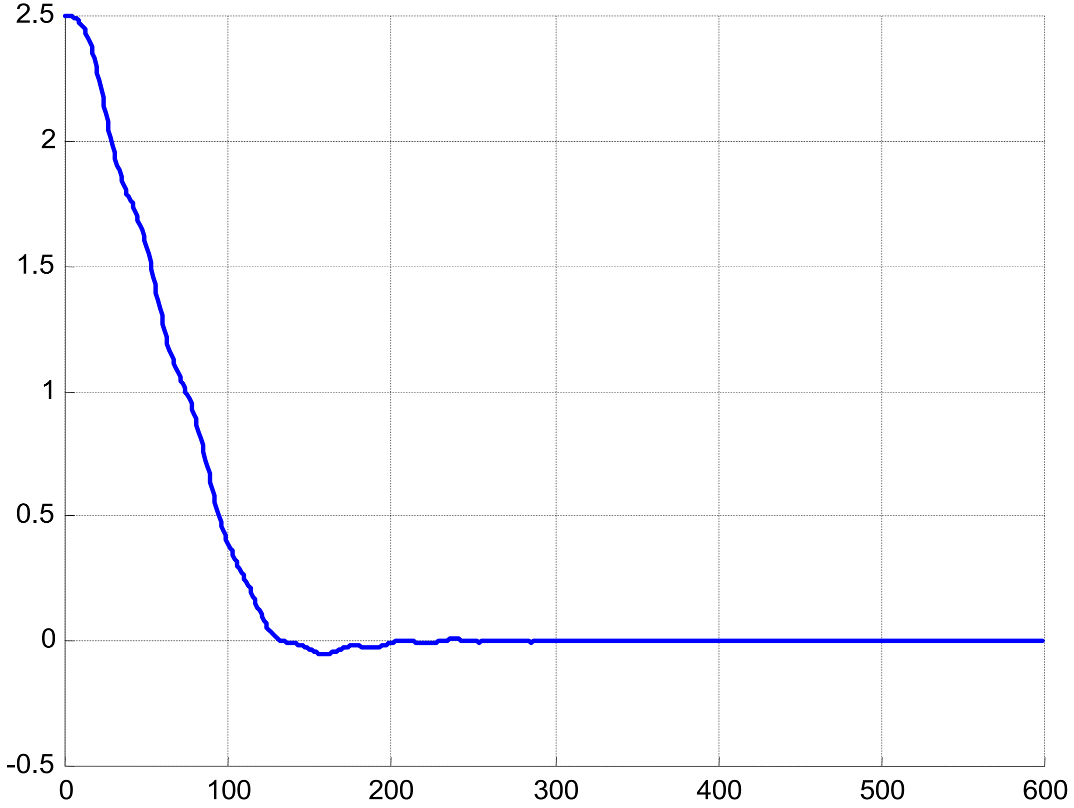
Disturbance $\boldsymbol{\theta}_1(\boldsymbol{t})$



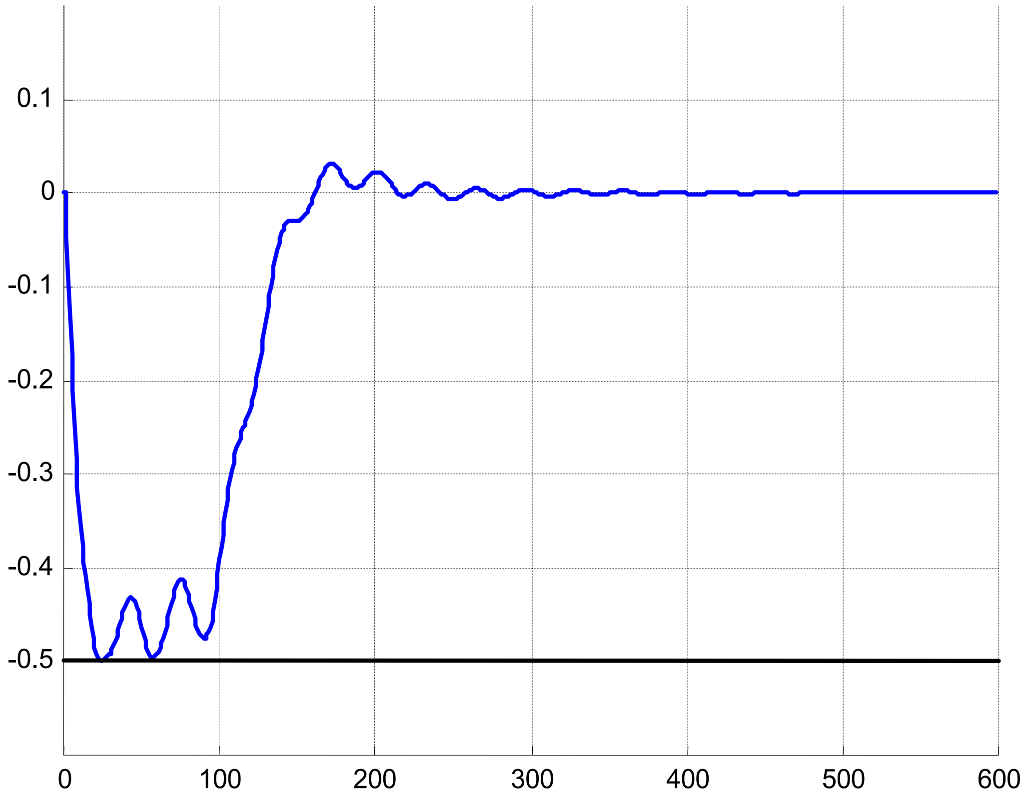
Disturbance $\boldsymbol{\theta}_2(\boldsymbol{t})$



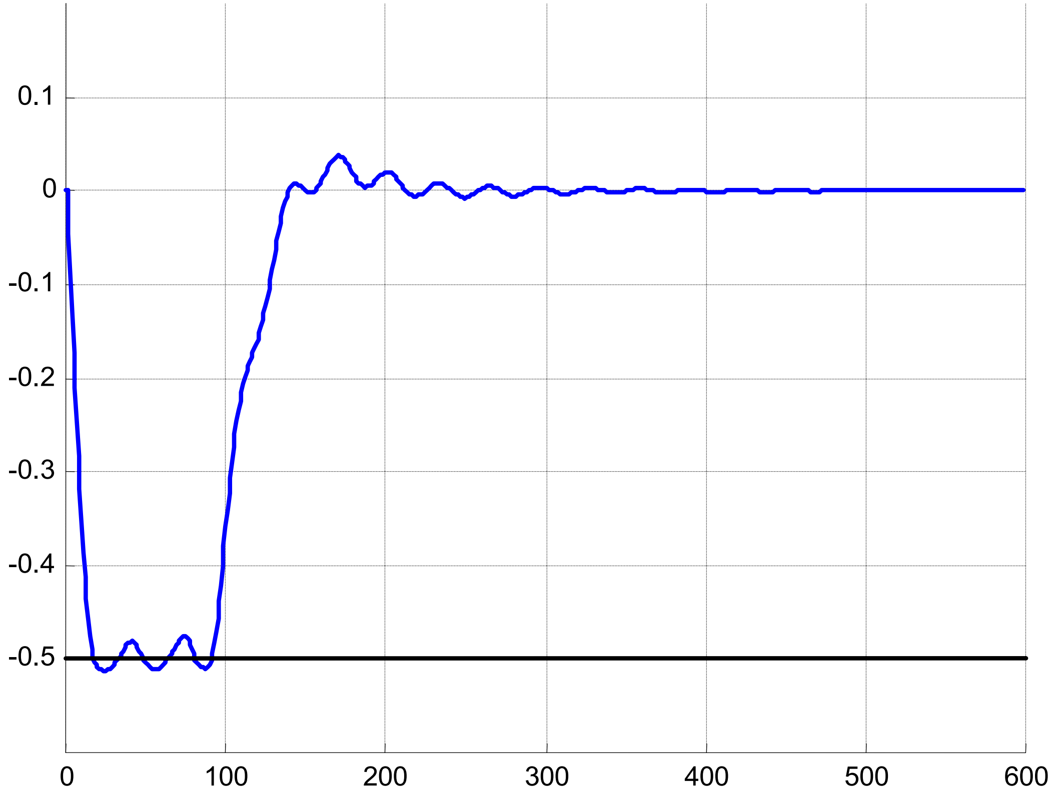
Control input trajectory with the <u>robust</u> MPC



State $\boldsymbol{x}_1$ trajectory with the robust MPC



State $\boldsymbol{x}_2$ trajectory with the robust MPC

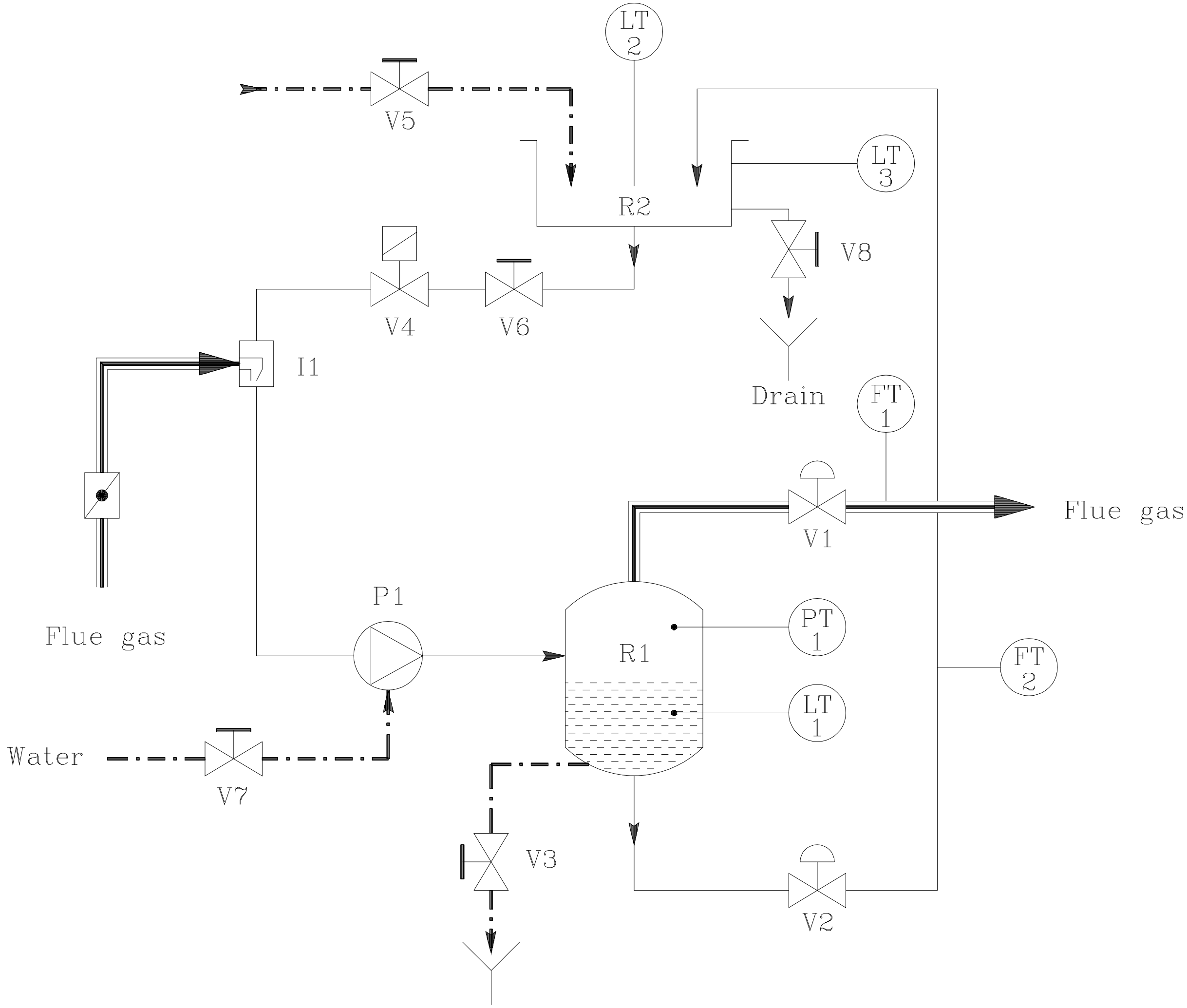


State $\boldsymbol{x}_2$ trajectory with the <u>nominal</u> MPC

6. Model Predictive Control of a Laboratory Separator

(Grancharova, Johansen, and Kocijan, 2004)

Pilot installation at Jozef Stefan Institute, Slovenia



Gas-liquid separator (Vrančić et al., 1995)

Process model: (Vrančić et al., 1995)

Valve V_1

$$K_1 = K_{01} \cdot R_{V1}{}^{v_1 - 1} = 75.1 \cdot 46.1^{v_1 - 1} \;\; [l/(\mathrm{s}\sqrt{\mathrm{bar}})]$$

$$v_1 = \begin{cases} 1 & if \;\; r_1 > 1 \\ 0 & if \;\; r_1 < 0 \\ r_1 & \text{otherwise} \end{cases}$$

Valve V_2

$$K_2 = K_{02} \cdot R_{V2}{}^{v_2 - 1} = 0.742 \cdot 75.66^{v_2 - 1} \;\; [l/(\mathrm{s}\sqrt{\mathrm{bar}})]$$

$$v_2 = \begin{cases} r_{2\max} & if \;\; r_2 > r_{2\max} \\ 0 & if \;\; r_2 < 0 \\ r_2 & \text{otherwise} \end{cases}$$

$$r_{2\max} = 0.8625$$

Speed limit of valves positions:

$$\dot{v} = \begin{cases} \dot{v}_{\max} & if \ \ \dot{r} > \dot{v}_{\max} \\ \dot{v}_{\min} & if \ \ \dot{r} < \dot{v}_{\min} \\ \dot{r} & \text{otherwise} \end{cases}$$

$$\dot{v}_{\max} = 0.66[\mathrm{s}^{-1}]$$

$$\dot{v}_{\min} = -0.33[\mathrm{s}^{-1}]$$

Air flow through valve V_1:

$$\Phi_1 = K_1 \sqrt{p_1} \ \ [l/\mathrm{s}]$$

Water flow through valve V_2:

$$\Phi_2 = K_2 \sqrt{p_1 + K_W \left(h_1 - h_{R2}\right)} \ \ [l/\mathrm{s}]$$

Water flow to the separator R_1:

$$\Phi_w = 0.1644 \ \ [l/\mathrm{s}]$$

Air flow to the separator R_1:

$$\Phi_{air} = \Phi_{air0} + \Phi_{air1} \cdot p_1 = 6.46 - 1.615 \cdot p_1 \ \ [l/\mathrm{s}]$$

The change of water level in R_1:

$$\frac{dh_1}{dt} = \frac{1}{S_1}\left(\Phi_w - \Phi_2\right)K_F \ \text{[m/s]}$$

The change of water level in R_2:

$$\frac{dh_2}{dt} = \frac{1}{S_2}\left(\Phi_2 - \Phi_w\right)K_F \ \text{[m/s]}$$

The change of air pressure inside R_1:

$$\frac{dp_1}{dt} = \frac{1}{V}\left[p_0\left(\Phi_{air} - \Phi_1\right)K_F + \left(p_0 + p_1\right)\left(\Phi_w - \Phi_2\right)K_F\right] \text{[bar/s]}$$

The air volume inside R_1:

$$V = S_1\left(h_{R1} - h_1\right) = S_1\left(2.25 - h_1\right) \ [\text{m}^3]$$

Linearized model of the gas-liquid separation plant

$$\begin{bmatrix} \Delta \dot{p}_1 \\ \Delta \dot{h}_1 \end{bmatrix} = A_c \begin{bmatrix} \Delta p_1 \\ \Delta h_1 \end{bmatrix} + B_c \begin{bmatrix} \Delta v_1 \\ \Delta v_2 \end{bmatrix}$$

$\Delta p_1 = p_1 - p_{1s}$ - change of pressure from the steady state;

$\Delta h_1 = h_1 - h_{1s}$ - change of liquid level;

$\Delta v_1 = v_1 - v_{1s}$ - change of position of Valve 1;

$\Delta v_2 = v_2 - v_{2s}$ - change of position of Valve 2;

Elements of A_c: (Vrančić et al., 1995)

$$a_{11} = \frac{K_F}{V_s}\left[p_0\left(\Phi_{air1} - \frac{K_{1s}}{2\sqrt{p_{1s}}} \right) - \left(p_0 + p_{1s} \right)\frac{K_{2s}}{2\sqrt{p_{V2s}}} \right]$$

$$a_{12} = -\frac{K_F}{V_s}\left[\left(p_0 + p_{1s} \right)\frac{K_{2s}K_w}{2\sqrt{p_{V2s}}} \right]$$

$$a_{21} = -\frac{K_F}{S_1}\left[\frac{K_{2s}}{2\sqrt{p_{V2s}}} \right]$$

$$a_{22} = -\frac{K_F}{S_1}\left[\frac{K_{2s}K_w}{2\sqrt{p_{V2s}}} \right]$$

Elements of *B_c*: (Vrančić et al., 1995)

$$b_{11} = -\frac{K_F}{V_s} p_0 K_{1s} \sqrt{p_{1s}} \ln\left(R_{V1}\right)$$

$$b_{12} = -\frac{K_F}{V_s} \left(p_0 + p_{1s}\right) K_{2s} \sqrt{p_{V2s}} \ln\left(R_{V2}\right)$$

$$b_{21} = 0$$

$$b_{22} = -\frac{K_F}{S_1} K_{2s} \sqrt{p_{V2s}} \ln\left(R_{V2}\right)$$

Linear continuous-time model:

$$A_c = \begin{bmatrix} -0.0285 & -0.0001 \\ -0.0006 & -0.0001 \end{bmatrix}, \qquad B_c = \begin{bmatrix} -0.0843 & -0.0041 \\ 0 & -0.0023 \end{bmatrix}$$

corresponding to steady state:

$$\boldsymbol{p}_{1\boldsymbol{s}} = 0.5\,\boldsymbol{bar}\ ,\ \boldsymbol{h}_{1\boldsymbol{s}} = 1.4\,\boldsymbol{m}\ ,\ \boldsymbol{v}_{1\boldsymbol{s}} = 0.4152\ ,\ \boldsymbol{v}_{2\boldsymbol{s}} = 0.7462$$

Linear discrete-time model:

- sampling time T_s=1sec
- steady state:

$$p_{1s} = 0.5\,bar\ ,\ h_{1s} = 1.4\,m\ ,\ v_{1s} = 0.4152\ ,\ v_{2s} = 0.7462$$

$$x(t+1) = Ax(t) + Bu(t)$$

$$A = \begin{bmatrix} 0.9719 & -0.0001 \\ -0.0006 & 0.9999 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.0832 & -0.0041 \\ 0 & -0.0023 \end{bmatrix}$$

$$x_1 = p_1 - p_{1s}\ [bar]\ ,\quad x_2 = h_1 - h_{1s}\ [m]$$

$$u_1 = v_1 - v_{1s}\ ,\quad u_2 = v_2 - v_{2s}$$

Two more states are added to take into account the integral error:

$$x_3(t+1) = x_3(t) + T_s x_1(t)$$

$$x_4(t+1) = x_4(t) + T_s x_2(t)$$

$$A = \begin{bmatrix} 0.9719 & -0.0001 & 0 & 0 \\ -0.0006 & 0.9999 & 0 & 0 \\ T_s & 0 & 1 & 0 \\ 0 & T_s & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0832 & -0.0041 \\ 0 & -0.0023 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Constraints:

$$0 \le \boldsymbol{v}_1 \le 1$$

$$0 \le v_2 \le 0.8625$$

$$-0.33 \le \dot{\boldsymbol{v}}_1 \le 0.66$$

$$-0.33 \le \dot{\boldsymbol{v}}_2 \le 0.66$$

Constraints on control variables:

$$-0.4152 \le \boldsymbol{u}_1(\boldsymbol{t}+\boldsymbol{k}) \le 0.5848 \quad , \ \mathrm{k} = 0,1,\ldots,\mathrm{N}-1$$

$$-0.7462 \le \boldsymbol{u}_2(\boldsymbol{t}+\boldsymbol{k}) \le 0.1163 \quad , \ \mathrm{k} = 0,1,\ldots,\mathrm{N}-1$$

Constraints on the rate of change of control variables:

$$-0.33\boldsymbol{T}_s \le \boldsymbol{u}_1(\boldsymbol{t}+\boldsymbol{k}) - \boldsymbol{u}_1(\boldsymbol{t}+\boldsymbol{k}-1) \le 0.66\boldsymbol{T}_s \quad , \ \mathrm{k} = 1,2,\ldots,\mathrm{N}-1$$

$$-0.33\boldsymbol{T}_s \le \boldsymbol{u}_2(\boldsymbol{t}+\boldsymbol{k}) - \boldsymbol{u}_2(\boldsymbol{t}+\boldsymbol{k}-1) \le 0.66\boldsymbol{T}_s \quad , \ \mathrm{k} = 1,2,\ldots,\mathrm{N}-1$$

Horizon: $\boldsymbol{N}$=500

Control input parameterization:

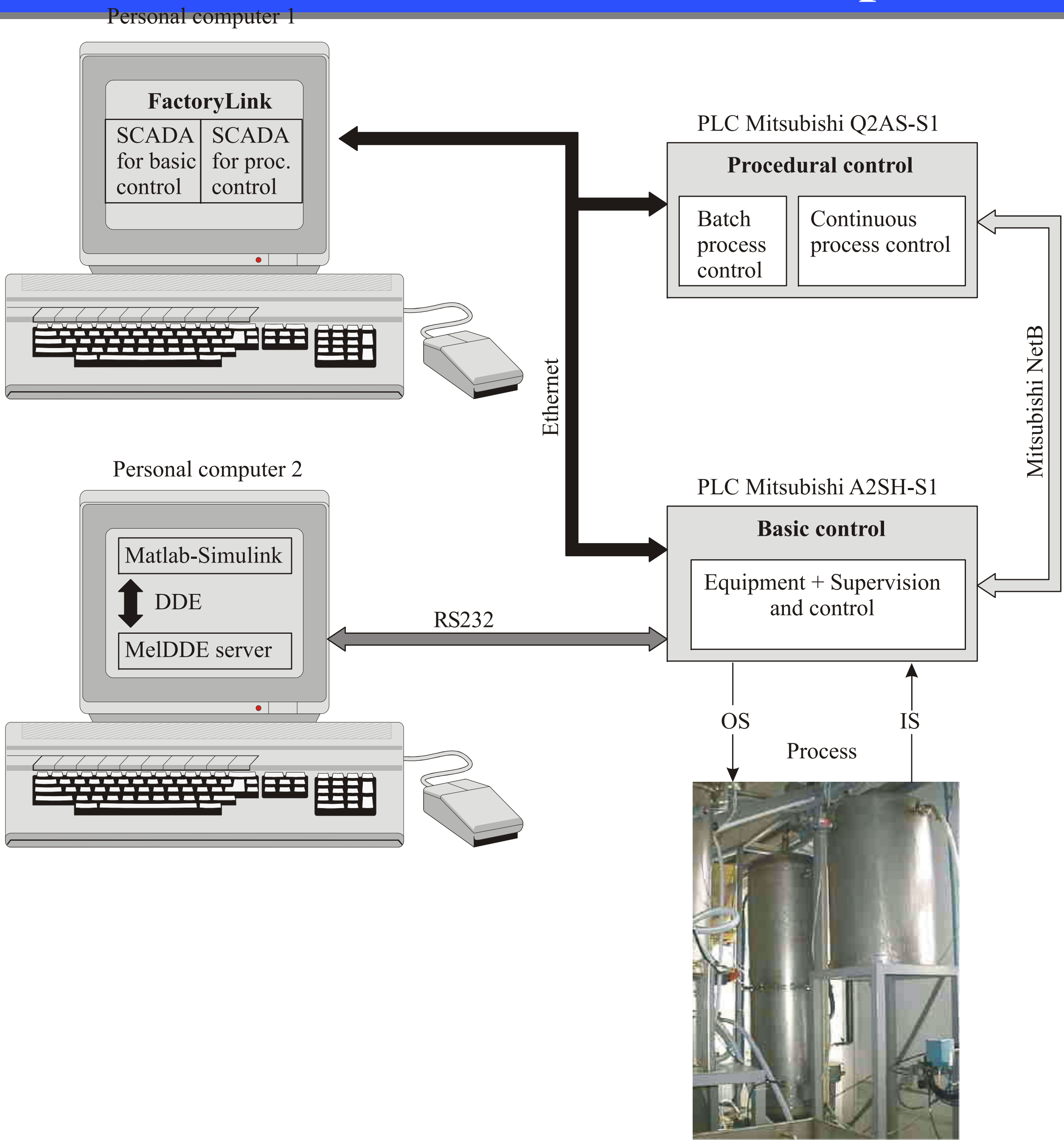
$$N_{u_1} = [1\ \ 5\ \ 10\ \ 15\ \ 20\ \ 25\ \ 30\ \ 35\ \ 40\ \ 45\ \ 50\ \ 100\ \ 102\ \ 104\ \ 106\ \ 108\ \ 110\ \ 300\ \ 302\ \ 304\ \ 306\ \ 308\ \ 310]$$

$$N_{u_2} = [1\ 5\ 10\ 15\ 20\ 25\ 30\ 35\ 40\ 45\ 50\ 100\ 300]$$

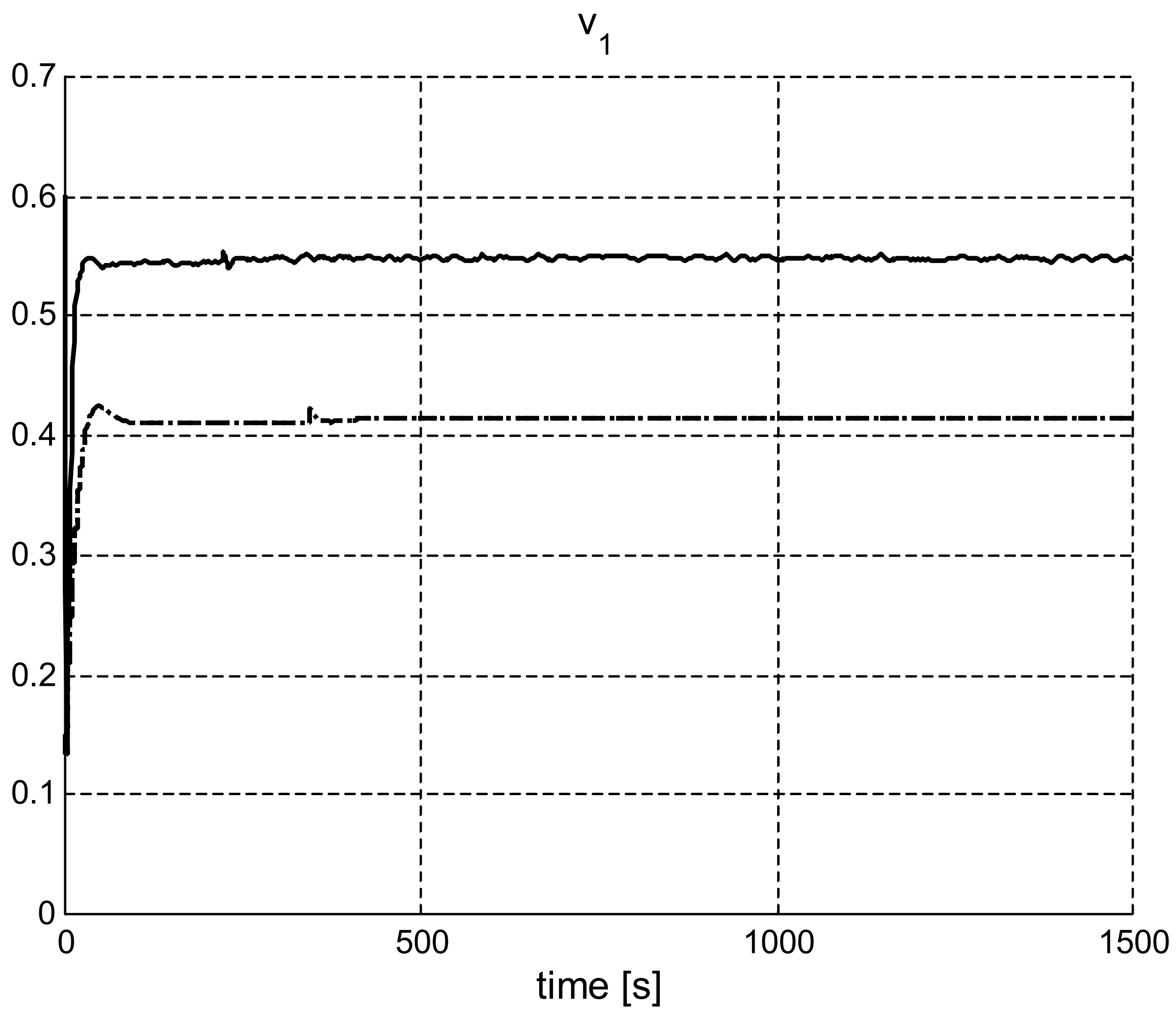
Cost matrices:

$\boldsymbol{Q}$=diag{0.05, 100, 0.005, 0.0001}, $\boldsymbol{R}$=diag{1, 1}

Environment for the real-time experiments



Real-time performance of the MPC controller

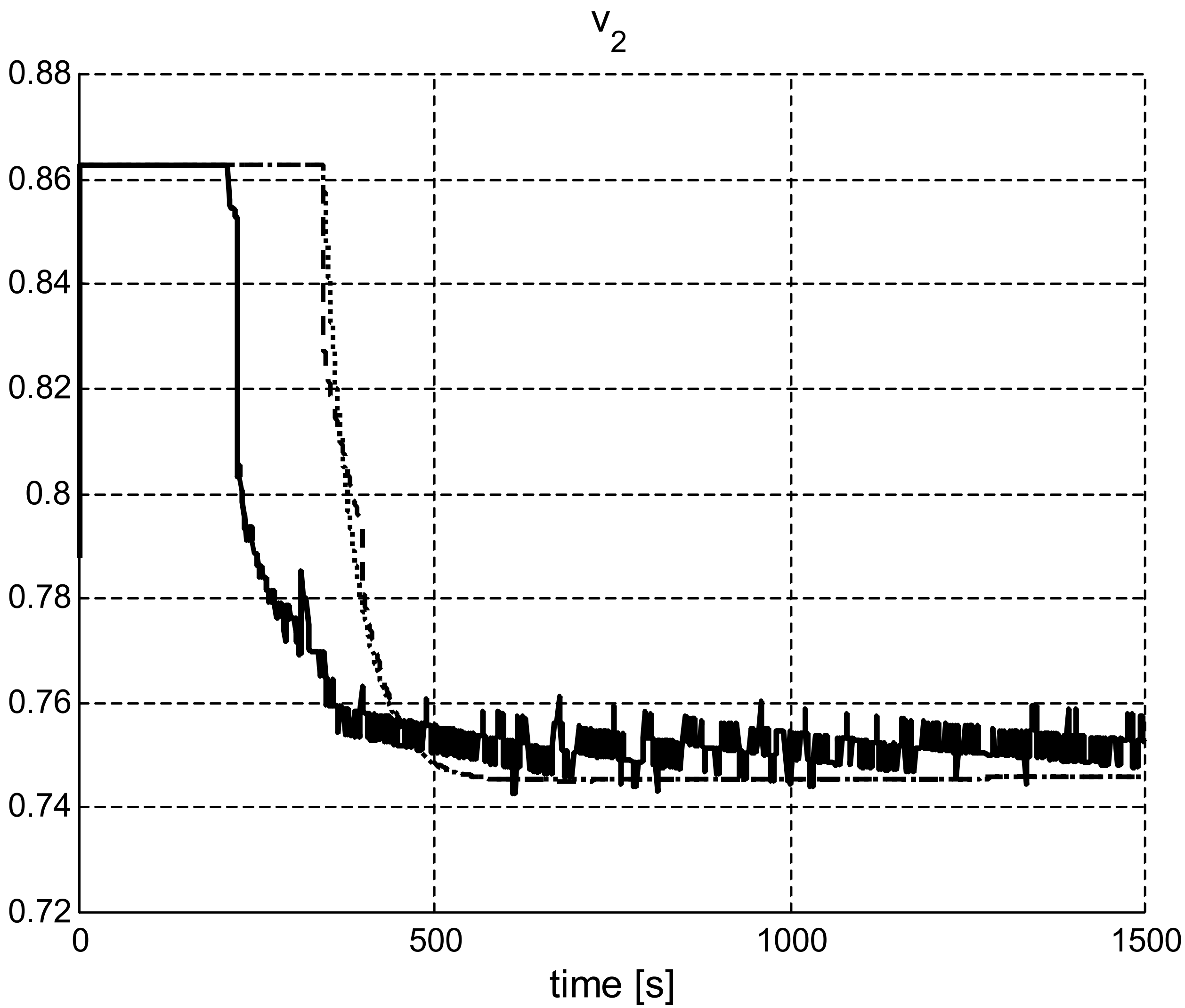


Valve 1

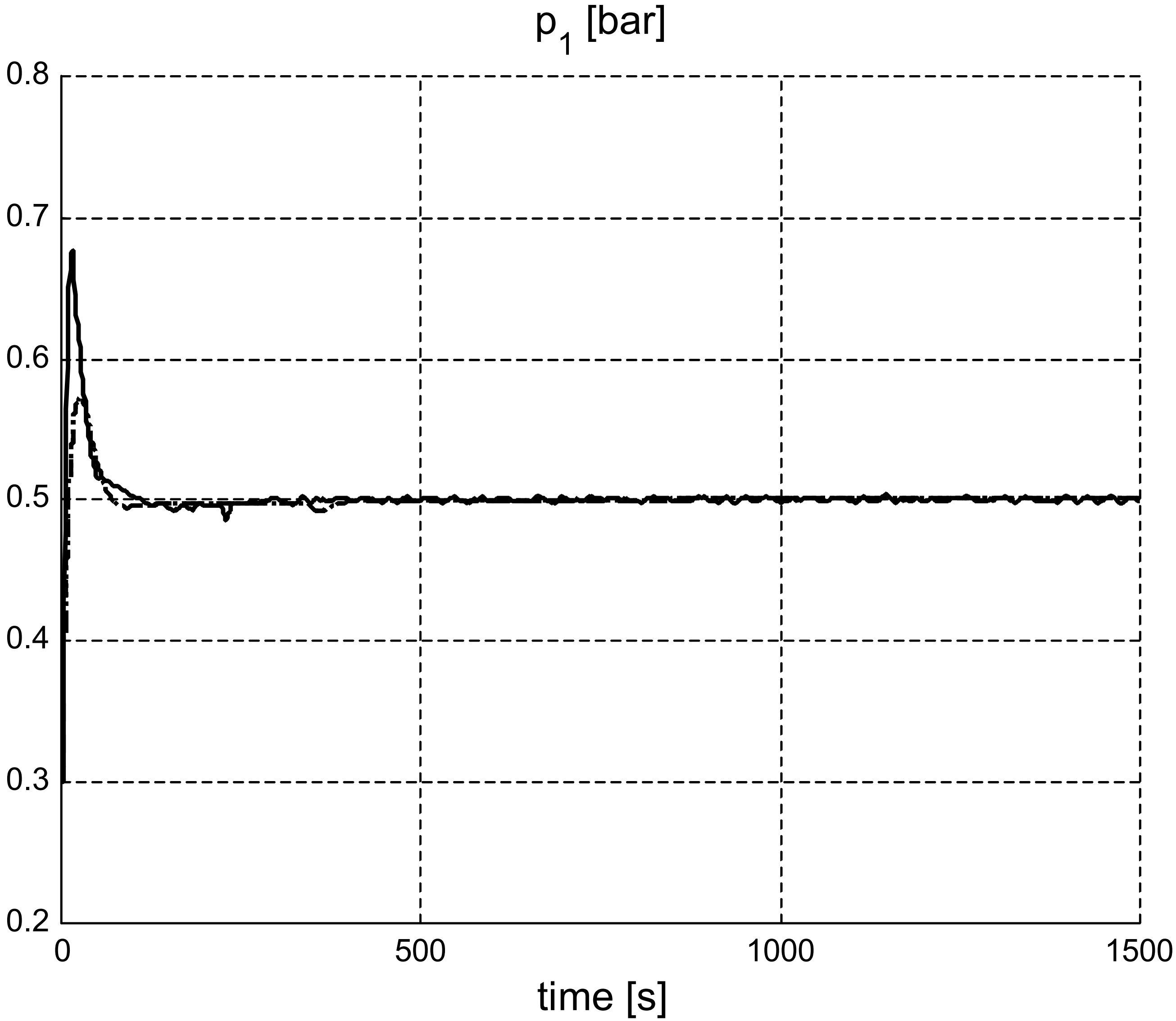
– experiments

… exact

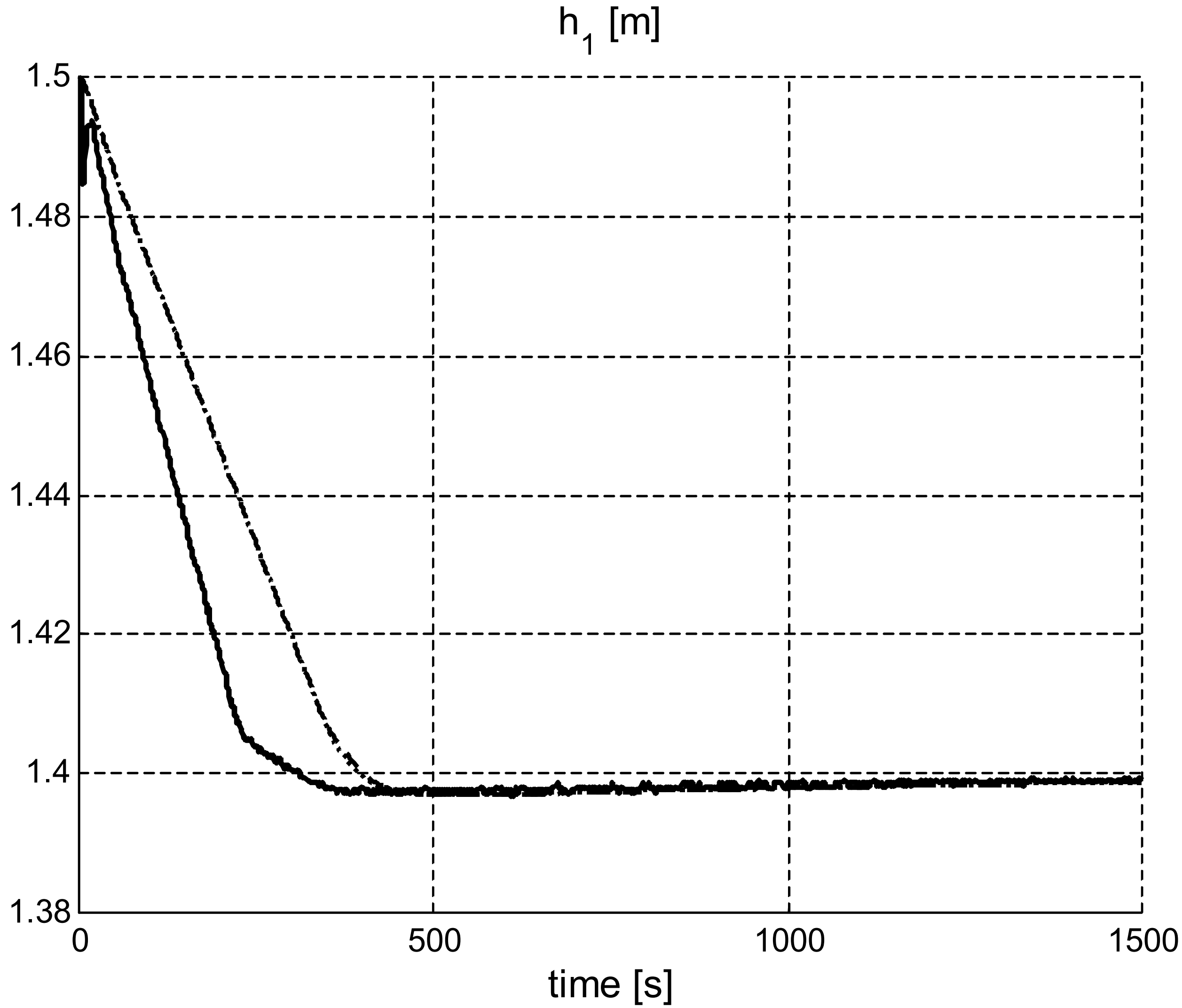
- - simulated



Valve 2 (— experiments, … exact, - - - simulated)

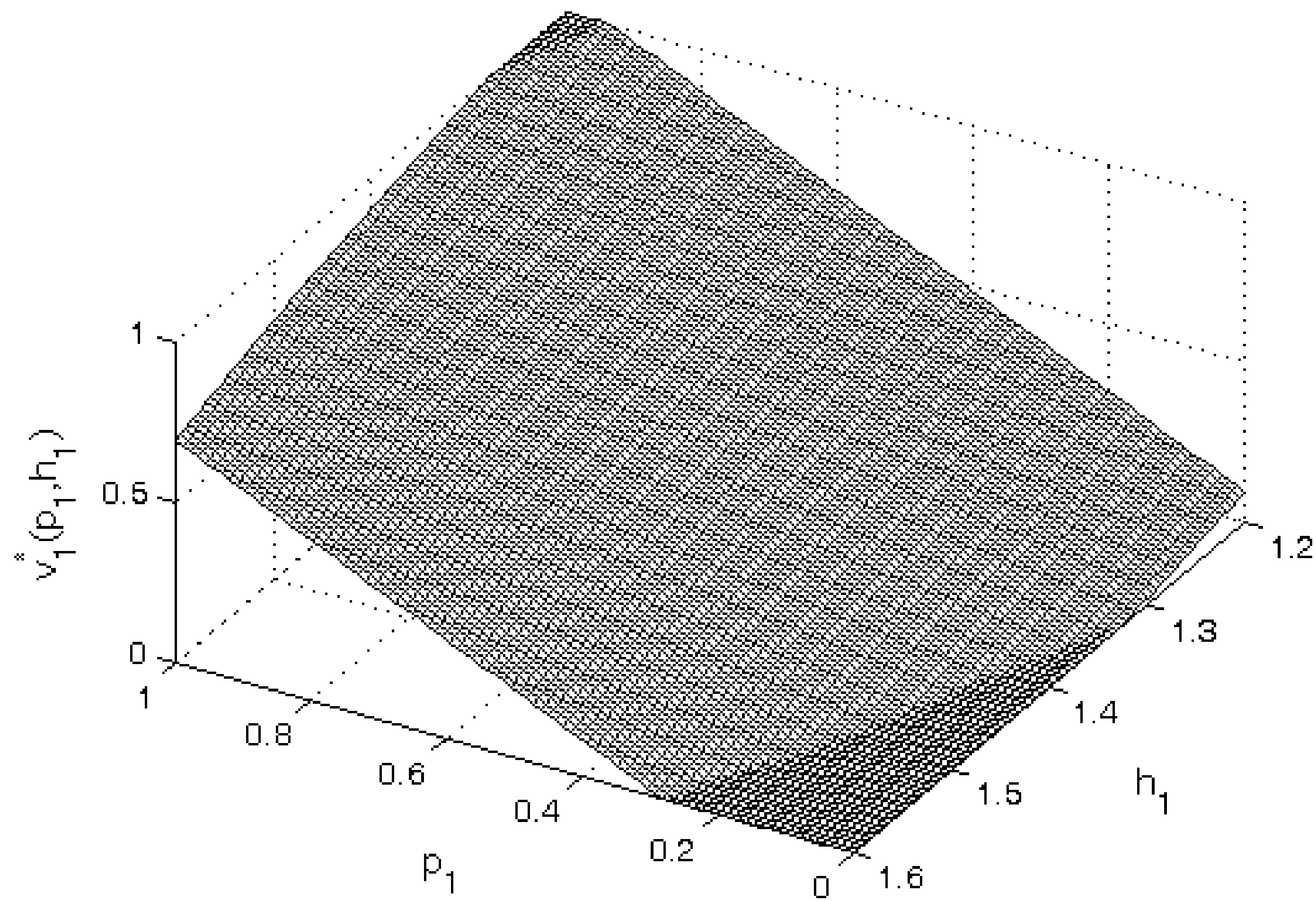


Pressure (— experiments, … exact, - - - simulated)



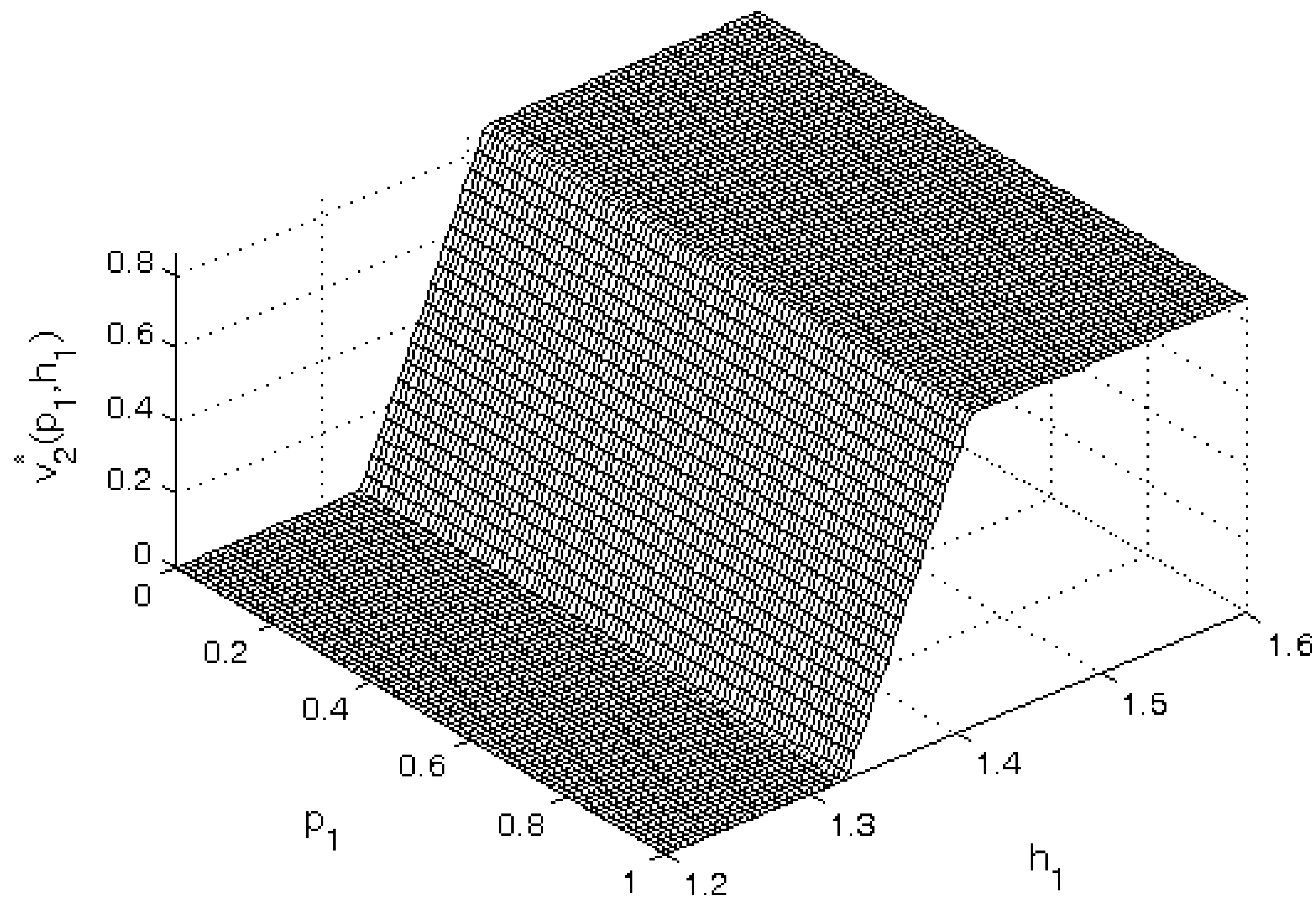
Liquid level (— experiments, … exact, - - - simulated)

Valve 1



Feedback law $v_1(p_1,h_1)$

Valve 2



Feedback law $v_2(p_1,h_1)$

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