# Ch. 3: Forward and Inverse Kinematics

ES159/259

# Updates

- Document clarifying the Denavit-Hartenberg convention is posted
- Labs and section times announced
  - If you haven't already, please forward your availability to Shelten & Ben
- Matlab review session Tuesday 2/13, 6:00 MD 221

# Recap: The Denavit-Hartenberg (DH) Convention

- Representing each individual homogeneous transformation as the product of four basic transformations:

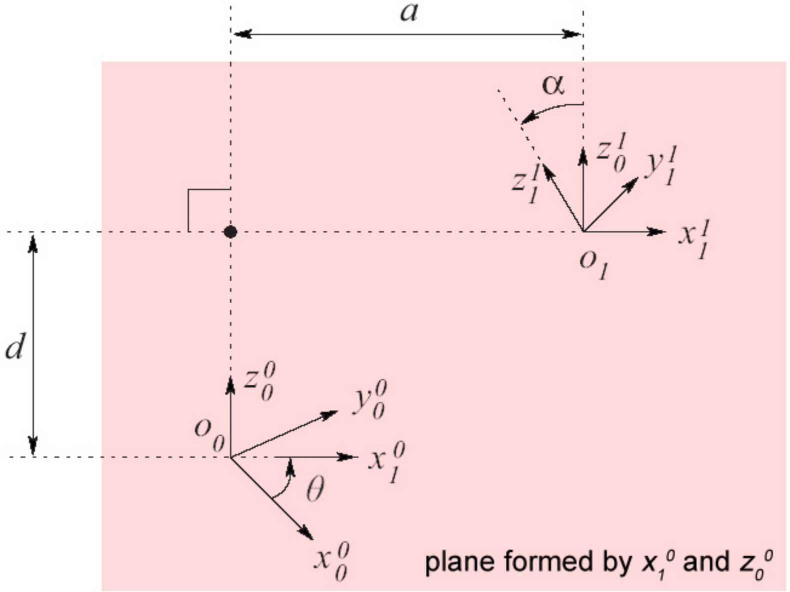
$$A_i = \mathbf{Rot}_{z,\theta_i}\mathbf{Trans}_{z,d_i}\mathbf{Trans}_{x,a_i}\mathbf{Rot}_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Recap: the physical basis for DH parameters

- $a_i$: link length, distance between the $o_0$ and $o_1$ (projected along $x_1$)
- $\alpha_i$: link twist, angle between $z_0$ and $z_1$ (measured around $x_1$)
- $d_i$: link offset, distance between $o_0$ and $o_1$ (projected along $z_0$)
- $\theta_i$: joint angle, angle between $x_0$ and $x_1$ (measured around $z_0$)

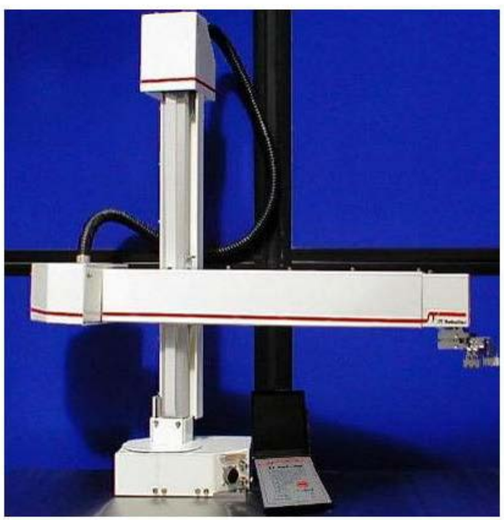


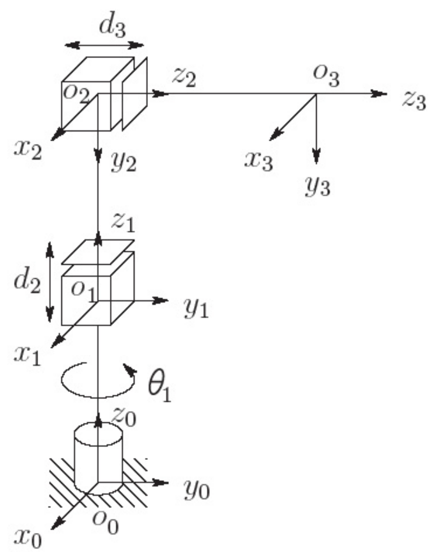
# General procedure for determining forward kinematics

1. Label joint axes as $z_0$, …, $z_{n-1}$ (axis $z_i$ is joint axis for joint $i$+1)
2. Choose base frame: set $o_0$ on $z_0$ and choose $x_0$ and $y_0$ using right-handed convention
3. For $i$=1:$n$-1,
   i. Place $o_i$ where the normal to $z_i$ and $z_{i-1}$ intersects $z_i$. If $z_i$ intersects $z_{i-1}$, put $o_i$ at intersection. If $z_i$ and $z_{i-1}$ are parallel, place $o_i$ along $z_i$ such that $d_i$=0
   ii. $x_i$ is the common normal through $o_i$, or normal to the plane formed by $z_{i-1}$ and $z_i$ if the two intersect
   iii. Determine $y_i$ using right-handed convention
4. Place the tool frame: set $z_n$ parallel to $z_{n-1}$
5. For $i$=1:$n$, fill in the table of DH parameters
6. Form homogeneous transformation matrices, $A_i$
7. Create $T_n^0$ that gives the position and orientation of the end-effector in the inertial frame

# Example 2: three-link cylindrical robot

- 3DOF: need to assign four coordinate frames
  1. Choose $z_0$ axis (axis of rotation for joint 1, base frame)
  2. Choose $z_1$ axis (axis of translation for joint 2)
  3. Choose $z_2$ axis (axis of translation for joint 3)
  4. Choose $z_3$ axis (tool frame)
     - This is again arbitrary for this case since we have described no wrist/gripper
     - Instead, define $z_3$ as parallel to $z_2$





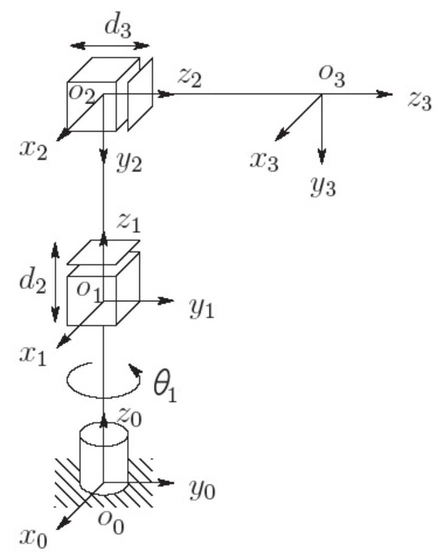
# Example 2: three-link cylindrical robot

- Now define DH parameters
  - First, define the constant parameters $a_i$, $\alpha_i$
  - Second, define the variable parameters $\theta_i$, $d_i$

| link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $d_1$ | $\theta_1$ |
| 2 | 0 | -90 | $d_2$ | 0 |
| 3 | 0 | 0 | $d_3$ | 0 |

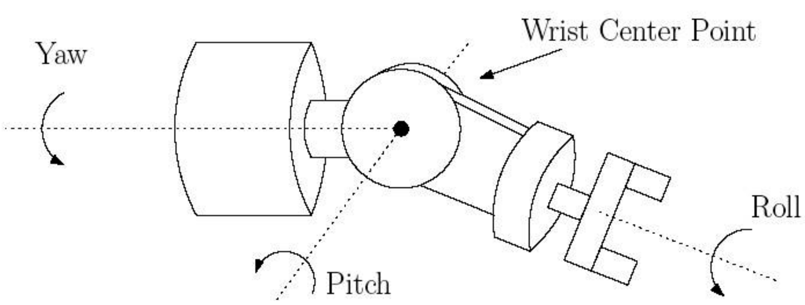
$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

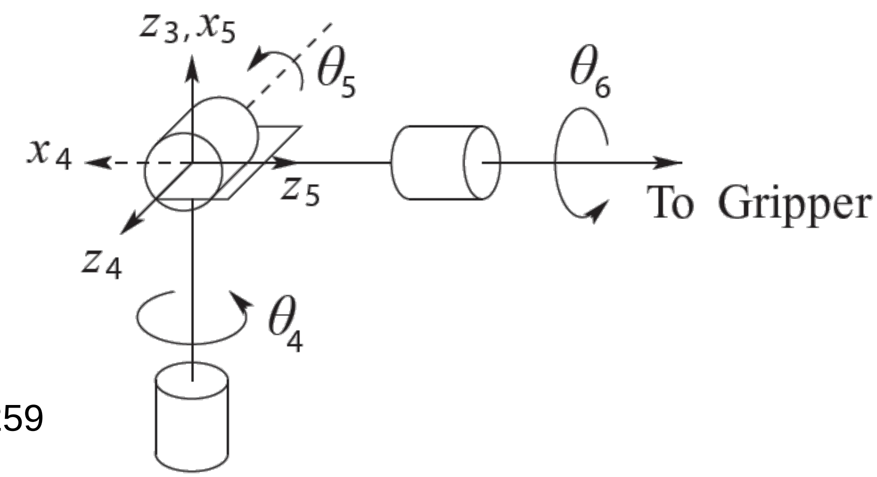
$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Example 3: spherical wrist

- 3DOF: need to assign four coordinate frames
  - yaw, pitch, roll ($\theta_4$, $\theta_5$, $\theta_6$) all intersecting at one point *o* (wrist center)
  1. Choose $z_3$ axis (axis of rotation for joint 4)
  2. Choose $z_4$ axis (axis of rotation for joint 5)
  3. Choose $z_5$ axis (axis of rotation for joint 6)
  4. Choose tool frame:
     - $z_6$ (*a*) is collinear with $z_5$
     - $y_6$ (*s*) is in the direction the gripper closes
     - $x_6$ (*n*) is chosen with a right-handed convention



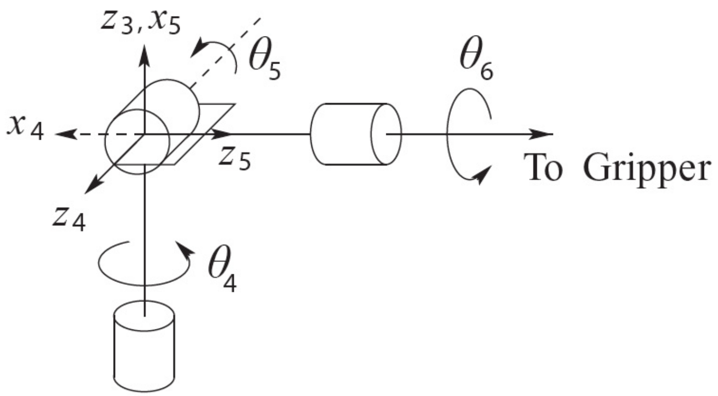
# Example 3: spherical wrist

- Now define DH parameters
  - First, define the constant parameters $a_i$, $\alpha_i$
  - Second, define the variable parameters $\theta_i$, $d_i$

| link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 4 | 0 | -90 | 0 | $\theta_4$ |
| 5 | 0 | 90 | 0 | $\theta_5$ |
| 6 | 0 | 0 | $d_6$ | $\theta_6$ |

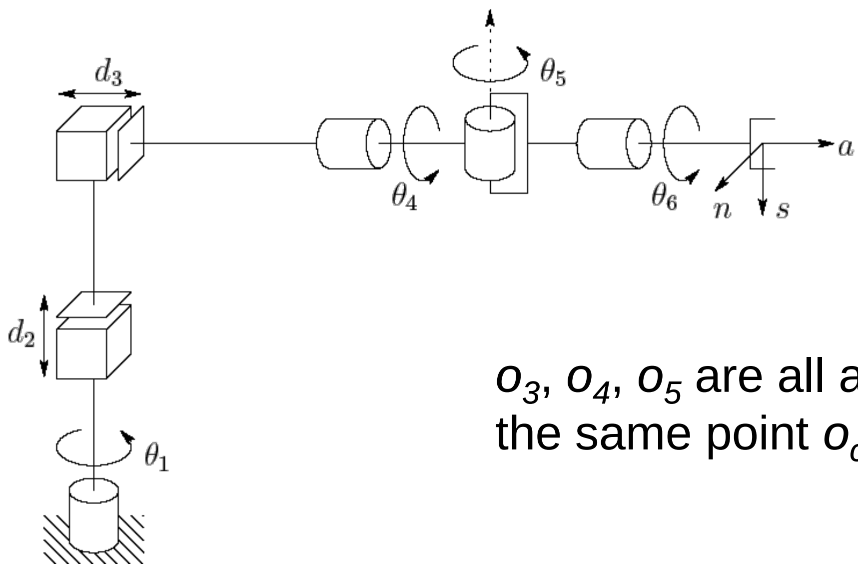
$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_5 = \begin{bmatrix} c_5 & 0 & -s_5 & 0 \\ s_5 & 0 & c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^3 = A_4 A_5 A_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5c_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Example 4: cylindrical robot with spherical wrist

- 6DOF: need to assign seven coordinate frames
  - But we already did this for the previous two examples, so we can fill in the table of DH parameters:



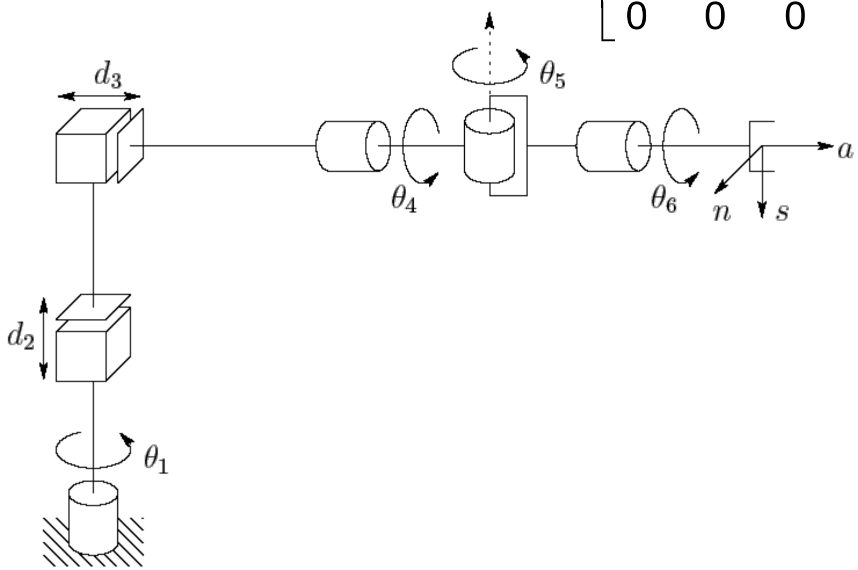

$o_3$, $o_4$, $o_5$ are all at the same point $o_c$

| link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $d_1$ | $\theta_1$ |
| 2 | 0 | -90 | $d_2$ | 0 |
| 3 | 0 | 0 | $d_3$ | 0 |
| 4 | 0 | -90 | 0 | $\theta_4$ |
| 5 | 0 | 90 | 0 | $\theta_5$ |
| 6 | 0 | 0 | $d_6$ | $\theta_6$ |

# Example 4: cylindrical robot with spherical wrist

- Note that $z_3$ (axis for joint 4) is collinear with $z_2$ (axis for joint 3), thus we can make the following combination:

$$T_6^0 = T_3^0 T_6^3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

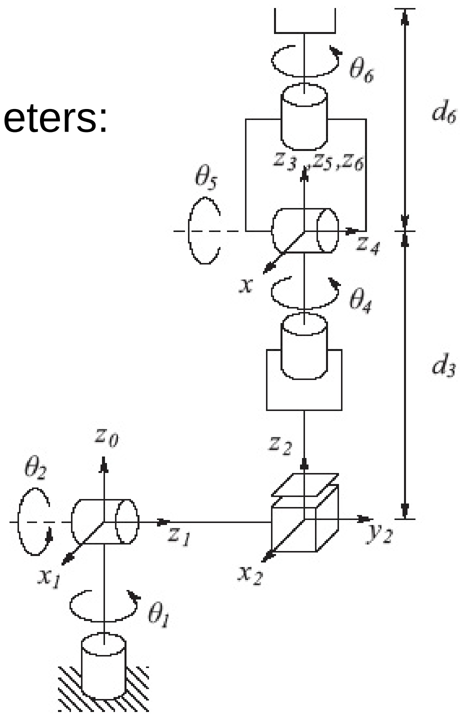


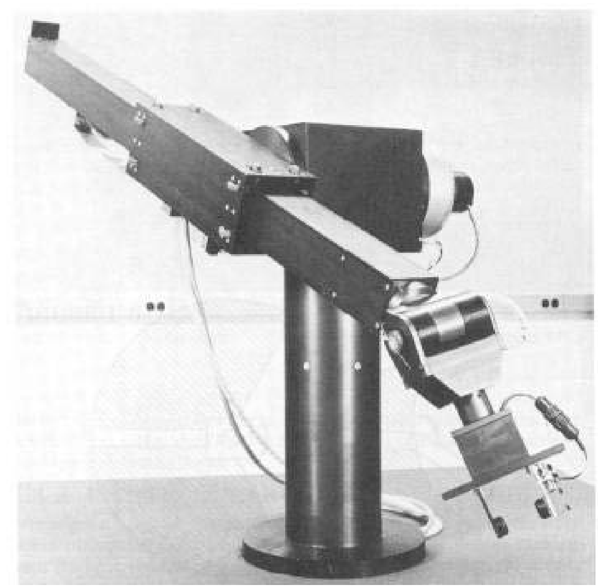

$$\begin{cases} r_{11} = c_1c_4c_5c_6 - c_1s_4s_6 + s_1s_5c_6 \\ r_{21} = s_1c_4c_5c_6 - s_1s_4s_6 - c_1s_5c_6 \\ r_{31} = -s_4c_5c_6 - c_4s_6 \\ r_{12} = -c_1c_4c_5s_6 - c_1s_4c_6 - s_1s_5c_6 \\ r_{22} = -s_1c_4c_5s_6 - s_1s_4s_6 + c_1s_5c_6 \\ r_{32} = s_4c_5c_6 - c_4c_6 \\ r_{13} = c_1c_4s_5 - s_1c_5 \\ r_{23} = s_1c_4s_5 + c_1c_5 \\ r_{33} = -s_4s_5 \\ d_x = c_1c_4s_5d_6 - s_1c_5d_6 - s_1d_3 \\ d_y = s_1c_4s_5d_6 + c_1c_5d_6 + c_1d_3 \\ d_z = -s_4s_5d_6 + d_1 + d_2 \end{cases}$$

# Example 5: the Stanford manipulator

- 6DOF: need to assign seven coordinate frames:
  1. Choose $z_0$ axis (axis of rotation for joint 1, base frame)
  2. Choose $z_1$-$z_5$ axes (axes of rotation/translation for joints 2-6)
  3. Choose $x_i$ axes
  4. Choose tool frame
  5. Fill in table of DH parameters:

| link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | -90 | 0 | $\theta_1$ |
| 2 | 0 | 90 | $d_2$ | $\theta_2$ |
| 3 | 0 | 0 | $d_3$ | 0 |
| 4 | 0 | -90 | 0 | $\theta_4$ |
| 5 | 0 | 90 | 0 | $\theta_5$ |
| 6 | 0 | 0 | $d_6$ | $\theta_6$ |



# Example 5: the Stanford manipulator

- Now determine the individual homogeneous transformations:

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Example 5: the Stanford manipulator

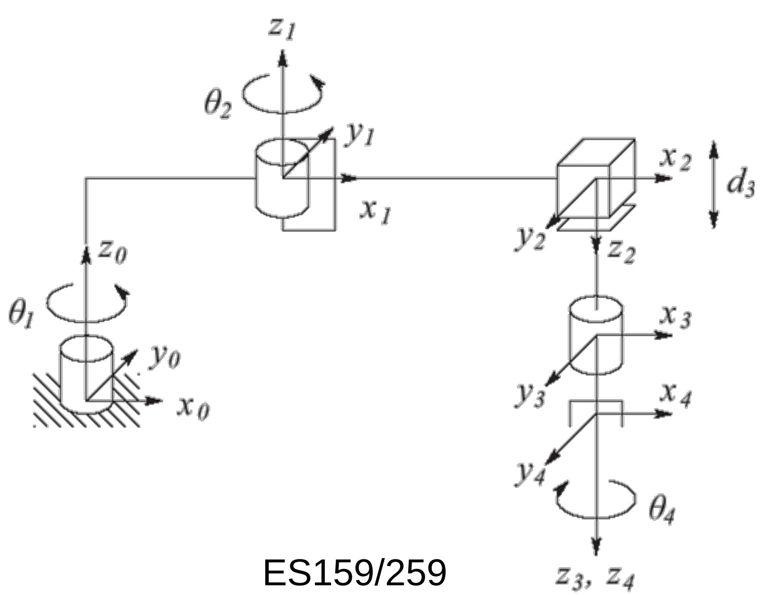
- Finally, combine to give the complete description of the forward kinematics:

$$T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \left\{ \begin{aligned}
r_{11} &= c_1\left[c_2\left(c_4c_5c_6 - s_4s_6\right) - s_2s_5c_6\right] - d_2\left(s_4c_5c_6 + c_4s_6\right) \\
r_{21} &= s_1\left[c_2\left(c_4c_5c_6 - s_4s_6\right) - s_2s_5c_6\right] + c_1\left(s_4c_5c_6 + c_4s_6\right) \\
r_{31} &= -s_2\left(c_4c_5c_6 - s_4s_6\right) - c_2s_5c_6 \\
r_{12} &= c_1\left[-c_2\left(c_4c_5s_6 + s_4c_6\right) + s_2s_5s_6\right] - s_1\left(-s_4c_5s_6 + c_4c_6\right) \\
r_{22} &= -s_1\left[-c_2\left(c_4c_5s_6 - s_4c_6\right) - s_2s_5s_6\right] + c_1\left(-s_4c_5s_6 + c_4s_6\right) \\
r_{32} &= s_2\left(c_4c_5s_6 + s_4c_6\right) + c_2s_5s_6 \\
r_{13} &= c_1\left(c_2c_4s_5 + s_2c_5\right) - s_1s_4s_5 \\
r_{23} &= s_1\left(c_2c_4s_5 + s_2c_5\right) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + d_6\left(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5\right) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6\left(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2\right) \\
d_z &= c_2d_3 + d_6\left(c_2c_5 - c_4s_2s_5\right)
\end{aligned} \right.$$

# Example 6: the SCARA manipulator

- 4DOF: need to assign five coordinate frames:
  1. Choose $z_0$ axis (axis of rotation for joint 1, base frame)
  2. Choose $z_1$-$z_3$ axes (axes of rotation/translation for joints 2-4)
  3. Choose $x_i$ axes
  4. Choose tool frame
  5. Fill in table of DH parameters:

| link | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | $a_1$ | 0 | 0 | $\theta_1$ |
| 2 | $a_2$ | 180 | 0 | $\theta_2$ |
| 3 | 0 | 0 | $d_3$ | 0 |
| 4 | 0 | 0 | $d_4$ | $\theta_4$ |




# Example 6: the SCARA manipulator

- Now determine the individual homogeneous transformations:

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Forward kinematics of parallel manipulators

- Parallel manipulator: two or more series chains connect the end-effector to the base (closed-chain)
- # of DOF for a parallel manipulator determined by taking the total DOFs for all links and subtracting the number of constraints imposed by the closed-chain configuration
- *Gruebler's formula* (3D):

$$\#\textbf{DOF} = 6\left(n_L - n_j\right) + \sum_{i=1}^{n_j} f_i$$

$n_L$: number of links*

$n_j$: number of joints

$f_i$: #DOF for joint *i*



*excluding ground

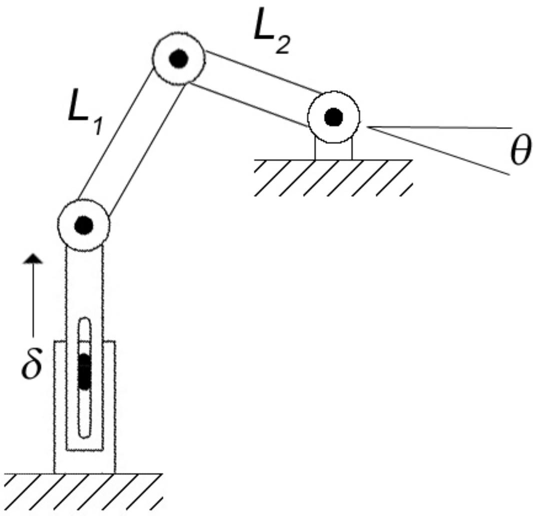
# Forward kinematics of parallel manipulators

- *Gruebler's formula* (2D):

$$\#\mathbf{DOF} = 3(n_L - n_j) + \sum_{i=1}^{n_j} f_i$$

- Example (2D):
  - Planar four-bar, $n_L = 3$, $n_j = 4$, $f_i = 1$(for all joints)
    - 3(3-4)+4 = 1DOF
  - Forward kinematics:

$$\theta = \cos^{-1}\left(\frac{\delta^2 - 2\delta + 2L_2^{\ 2}}{2\sqrt{(L_1-\delta)^2 + L_2^{\ 2}}}\right) + \tan^{-1}\left(\frac{L_2}{L_1-\delta}\right) - \frac{\pi}{2}$$



# Inverse Kinematics

- Find the values of joint parameters that will put the tool frame at a desired position and orientation (within the workspace)
  - Given *H*:

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3)$$

  - Find *all* solutions to:

$$T_n^0(q_1, \ldots, q_n) = H$$

  - Noting that:

$$T_n^0(q_1, \ldots, q_n) = A_1(q_1) \cdots A_n(q_n)$$

  - This gives 12 (nontrivial) equations with *n* unknowns

# Example: the Stanford manipulator

- For a given $H$:

$$H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

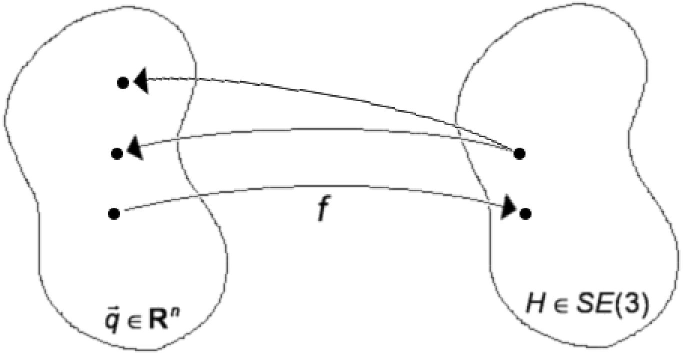
- Find $\theta_1$, $\theta_2$, $d_3$, $\theta_4$, $\theta_5$, $\theta_6$:

$$\underbrace{\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
-s_1[-c_2(c_4c_5s_6 - s_4c_6) - s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4s_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}}$$

- One solution: $\theta_1 = \pi/2$, $\theta_2 = \pi/2$, $d_3 = 0.5$, $\theta_4 = \pi/2$, $\theta_5 = 0$, $\theta_6 = \pi/2$

# Inverse Kinematics

- The previous example shows how difficult it would be to obtain a closed-form solution to the 12 equations
- Instead, we develop systematic methods based upon the manipulator configuration
- For the forward kinematics there is always a unique solution
  - Potentially complex nonlinear functions
- The inverse kinematics may or may not have a solution
  - Solutions may or may not be unique
  - Solutions may violate joint limits
- Closed-form solutions are ideal!



# Overview: kinematic decoupling

- Appropriate for systems that have an arm a wrist
  - Such that the wrist joint axes are aligned at a point
- For such systems, we can split the inverse kinematics problem into two parts:
  1. Inverse position kinematics: position of the wrist center
  2. Inverse orientation kinematics: orientation of the wrist
- First, assume 6DOF, the last three intersecting at $o_c$

$$R_6^0(q_1,\ldots,q_6) = R$$
$$o_6^0(q_1,\ldots,q_6) = o$$

- Use the position of the wrist center to determine the first three joint angles…

# Overview: kinematic decoupling

- Now, origin of tool frame, $o_6$, is a distance $d_6$ translated along $z_5$ (since $z_5$ and $z_6$ are collinear)
  - Thus, the third column of $R$ is the direction of $z_6$ (w/ respect to the base frame) and we can write:

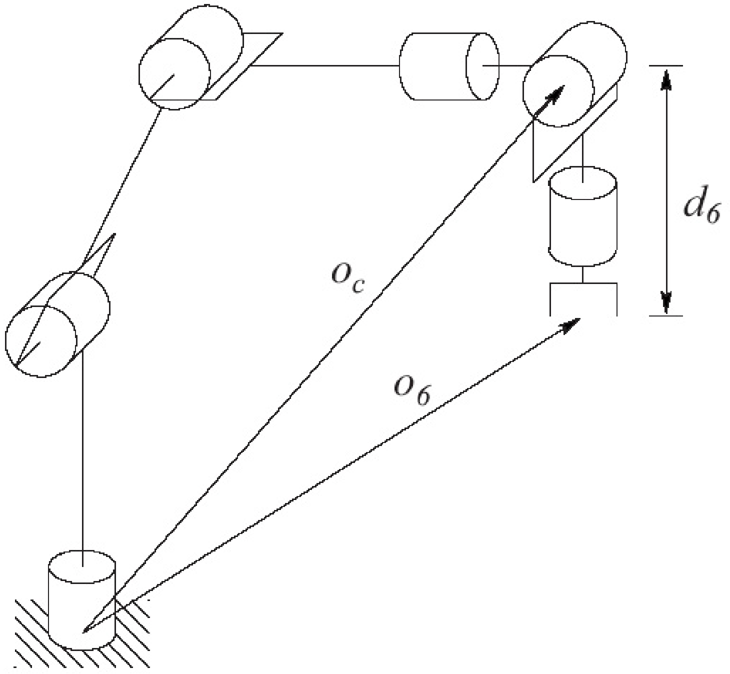
$$o = o_6^0 = o_c^o + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - Rearranging:

$$o_c^o = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - Calling $o = [o_x\ o_y\ o_z]^T$, $o_c^0 = [x_c\ y_c\ z_c]^T$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$



# Overview: kinematic decoupling

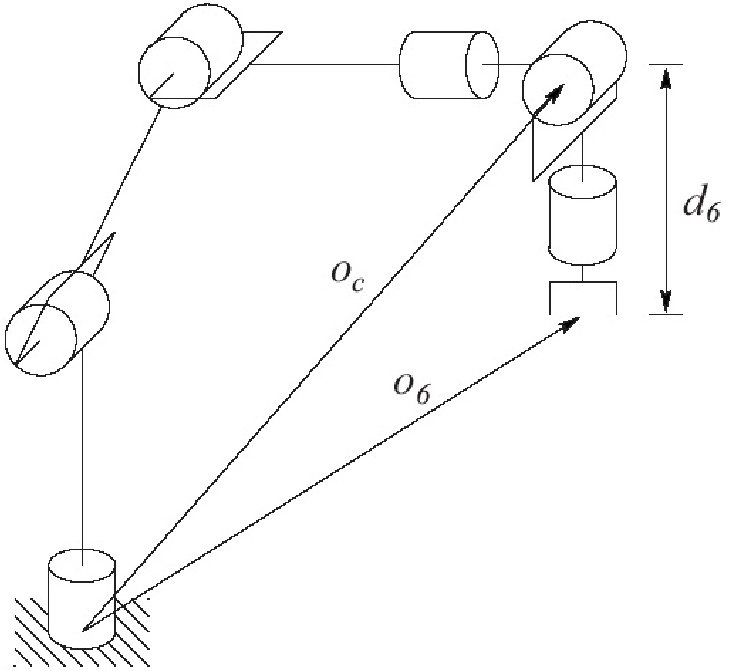
- Since $[x_c\ y_c\ z_c]^T$ are determined from the first three joint angles, our forward kinematics expression now allows us to solve for the first three joint angles decoupled from the final three.
  - Thus we now have $R_3^0$
  - Note that:

$$R = R_3^0 R_6^3$$

  - To solve for the final three joint angles:

$$R_6^3 = \left(R_3^0\right)^{-1} R = \left(R_3^0\right)^T R$$

  - Since the last three joints for a spherical wrist, we can use a set of Euler angles to solve for them



# Inverse position

- Now that we have $[x_c\ y_c\ z_c]^T$ we need to find $q_1$, $q_2$, $q_3$
  - Solve for $q_i$ by projecting onto the $x_{i-1}$, $y_{i-1}$ plane, solve trig problem
  - Two examples: elbow (RRR) and spherical (RRP) manipulators
  - For example, for an elbow manipulator, to solve for $\theta_1$, project the arm onto the $x_0$, $y_0$ plane

# Background: two argument atan

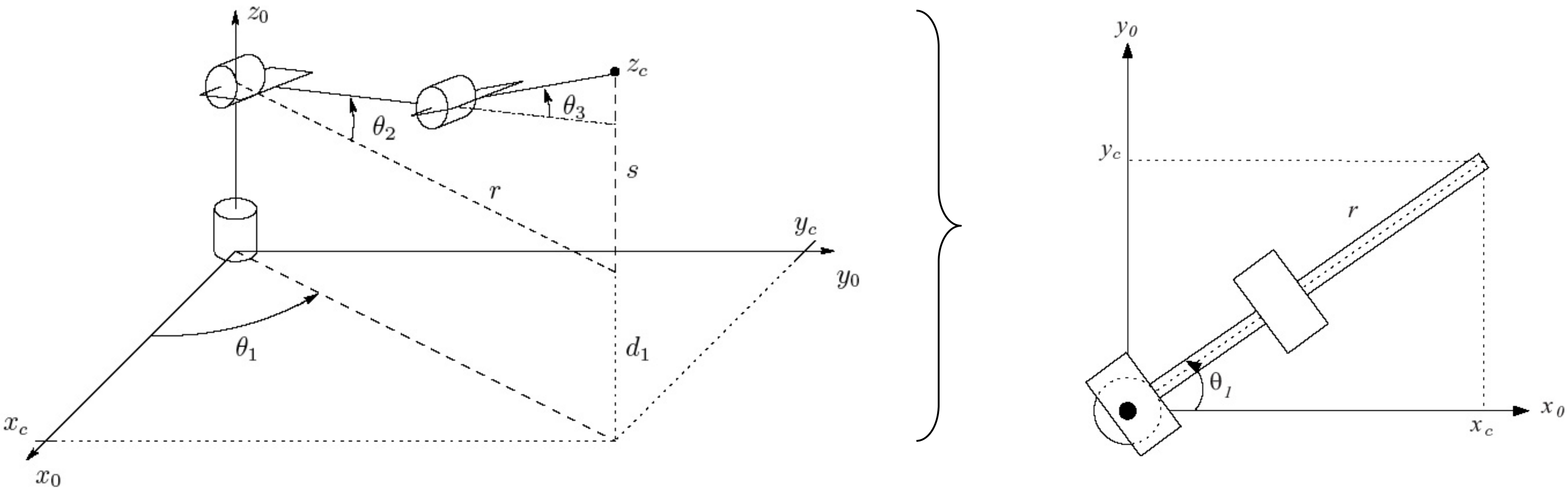
- We use atan2(·) instead of atan(·) to account for the full range of angular solutions
  - Called 'four-quadrant' arctan

$$\mathbf{atan}2(y,x)=\begin{cases} -\mathbf{atan}2(-y,x) & y<0 \\ \pi-\mathbf{atan}\left(-\dfrac{y}{x}\right) & y\geq 0, x<0 \\ \mathbf{atan}\left(\dfrac{y}{x}\right) & y\geq 0, x\geq 0 \\ \dfrac{\pi}{2} & y>0, x=0 \\ \textbf{undefined} & y=0, x=0 \end{cases}$$

# Example: RRR manipulator

1. To solve for $\theta_1$, project the arm onto the $x_0$, $y_0$ plane

$$\theta_1 = \mathbf{atan2}(x_c, y_c)$$

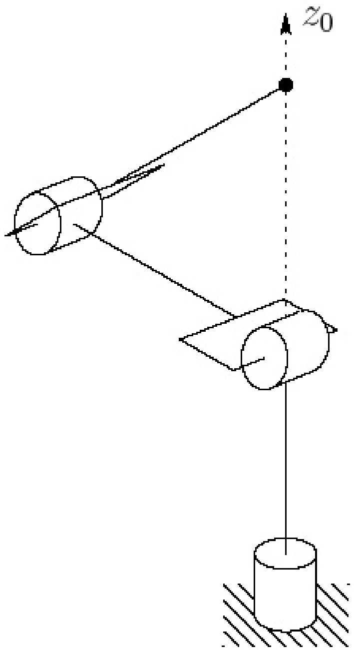



– Can also have: $\theta_1 = \pi + \mathbf{atan2}(x_c, y_c)$

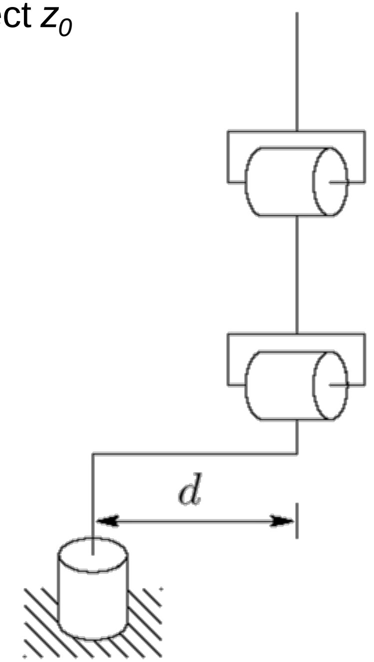
- This will of course change the solutions for $\theta_2$ and $\theta_3$

# Caveats: singular configurations, offsets

- If $x_c=y_c=0$, $\theta_1$ is undefined
  - i.e. any value of $\theta_1$ will work



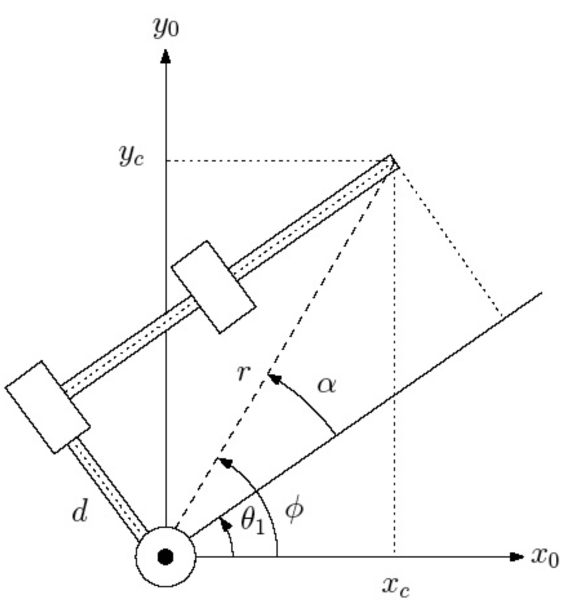

- If there is an offset, then we will have two solutions for $\theta_1$: *left arm* and *right arm*
  - However, wrist centers cannot intersect $z_0$



# Left arm and right arm solutions

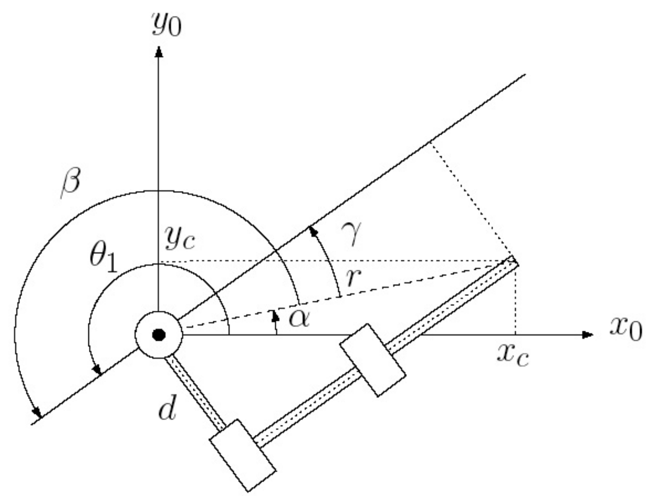
- Left arm:

$$\theta_1 = \phi - \alpha$$
$$\phi = \mathbf{atan2}(x_c, y_c)$$
$$\alpha = \mathbf{atan2}\left(\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, d\right)$$




- Right arm:

$$\theta_1 = \alpha + \beta$$
$$\alpha = \mathbf{atan2}(x_c, y_c)$$
$$\beta = \pi + \mathbf{atan2}\left(\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, d\right)$$
$$= \mathbf{atan2}\left(-\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, -d\right)$$



# Left arm and right arm solutions

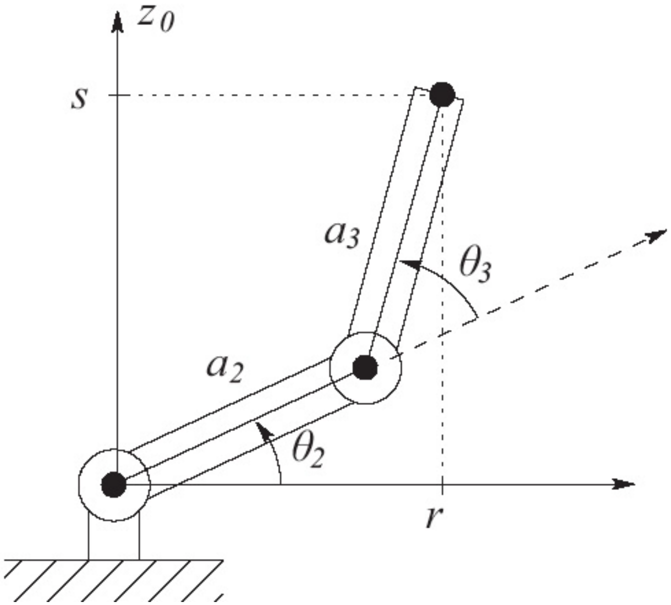
- Therefore there are in general two solutions for $\theta_1$
- Finding $\theta_2$ and $\theta_3$ is identical to the planar two-link manipulator we have seen previously:

$$\cos\theta_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

$$r^2 = x_c^2 + y_c^2 - d^2$$

$$s = z_c - d_1$$

$$\Rightarrow \cos\theta_3 = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} \equiv D$$



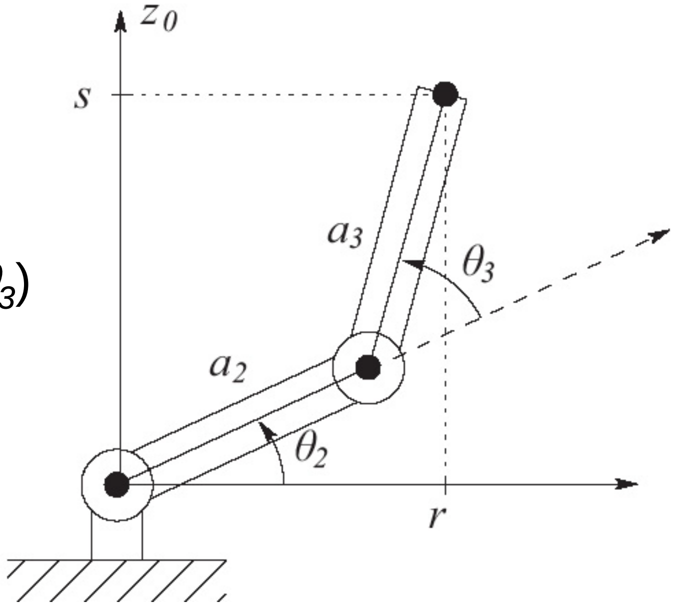

- Therefore we can find two solutions for $\theta_3$:

$$\theta_3 = \mathbf{atan2}\left(D, \pm\sqrt{1 - D^2}\right)$$

# Left arm and right arm solutions

- The two solutions for $\theta_3$ correspond to the elbow-down and elbow-up positions respectively
- Now solve for $\theta_2$:

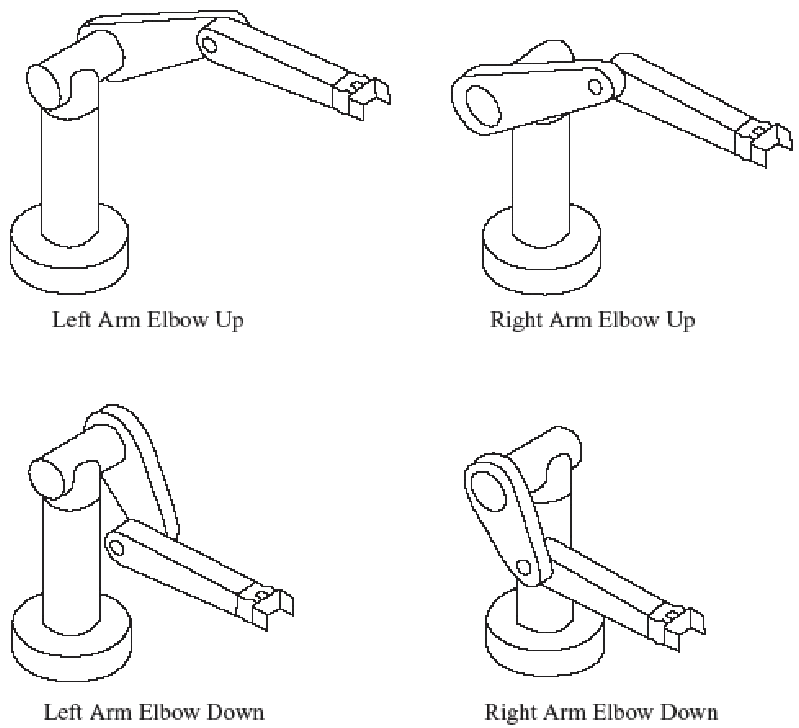
$$\theta_2 = \textbf{atan}2(r,s) - \textbf{atan}2(a_2 + a_3c_3, a_3s_3)$$
$$= \textbf{atan}2\left(\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, z_c - d_1\right) - \textbf{atan}2(a_2 + a_3c_3, a_3s_3)$$




- Thus there are two solutions for the pair ($\theta_2$, $\theta_3$)

# RRR: Four total solutions

- In general, there will be a maximum of four solutions to the inverse *position* kinematics of an elbow manipulator
  - Ex: PUMA



# Example: RRP manipulator

- Spherical configuration
  - Solve for $\theta_1$ using same method as with RRR

$$\theta_1 = \textbf{atan}2(x_c, y_c)$$

  - Again, if there is an offset, there will be left-arm and right-arm solutions
  - Solve for $\theta_2$:

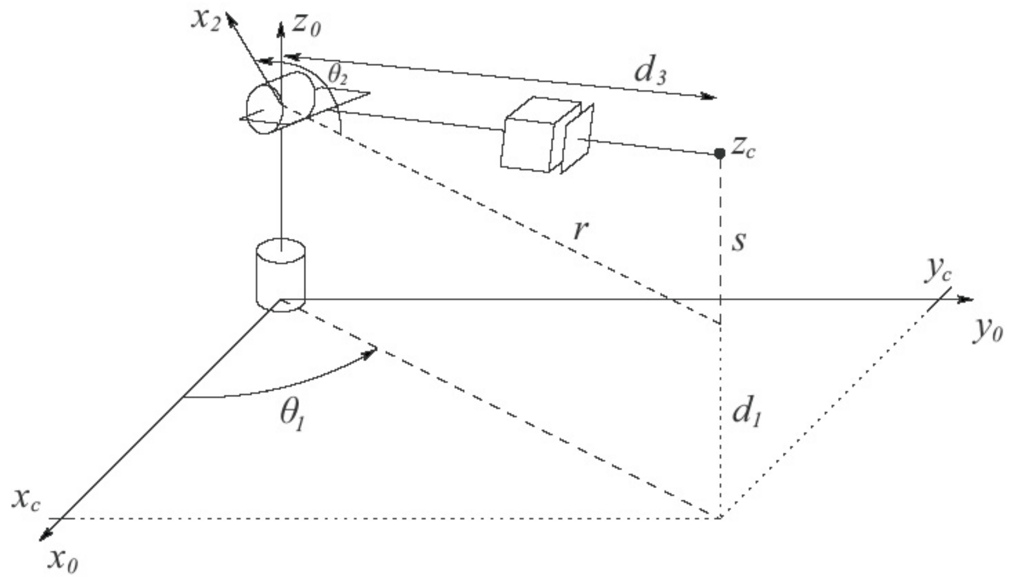
$$\theta_2 = \textbf{atan}2(s, r)$$

$$r^2 = x_c^{\ 2} + y_c^{\ 2}$$

$$s = z_c - d_1$$

  - Solve for $d_3$:

$$d_3 = \sqrt{r^2 + s^2} = \sqrt{x_c^{\ 2} + y_c^{\ 2} + (z_c - d_1)^2}$$



# Next class…

- Complete the discussion of inverse kinematics
  - Inverse orientation
  - Introduction to other methods
- Introduction to velocity kinematics and the Jacobian